Math 251 Exam 3 A Thursday, April 4, 2024

Printed Name: Key

Section:

UIN:

Signature: On my honor, as an Aggie, I have neither given nor received unauthorized aid on this academic work.

- **Show all appropriate work to receive full credit.**
- **You will be graded not merely on your final answer, but also on the quality and correctness of the work leading up to it.**
- If you need more space to work a problem, you may use the back of the cover page or the back of the exam. Please indicate where the problem is located.
- **Calculators are not allowed.**
- SCHOLASTIC DISHONESTY WILL NOT BE TOLERATED.

Good Luck!

1. (15 points) True or False. Circle your answer.

T F If a, b, c, and d are constants such that $0 < a < b < c < d$, then

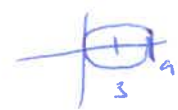
$$\int_{z=0}^{d} \int_{x=a}^{z+5} \int_{y=b}^{c} (2xy^2 + y^2z^2)\ dydxdz = \int_{y=b}^{c} y^2\ dy \int_{z=0}^{d} \int_{x=a}^{z+5} (2x + z^2)\ dxdz$$

T F The graph (in spherical coordinates) of the equation $\rho \cos(\theta) \sin(\phi) = 3$ is a plane.

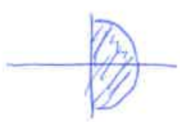
T F The graph (in polar coordinates) of the equation $r = 9\cos(\theta)$ is a circle of radius 9.

$r^2 = 9r\cos\theta$

$x^2+y^2 = 9x$



T F $\displaystyle\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} g(x,y)dydx = \int_0^2 \int_0^{\pi} g(r\cos(\theta), r\sin(\theta))rd\theta dr$



T F Region D is described by the part of the xy−plane such that $a \leq r \leq b$, $c \leq \theta \leq d$.

$\displaystyle\int_{\theta=c}^{d} \int_{r=a}^{b} r\ drd\theta$ is equivalent to the area of region D.

2. (6 points) Give the equation in spherical coordinates. Simplify your answer.

$z = -\sqrt{\frac{x^2}{3} + \frac{y^2}{3}}$ ← Lower part of the full cone.

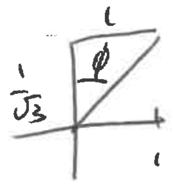
Consider $z = \sqrt{\frac{x^2}{3} + \frac{y^2}{3}}$

Answer: $\phi = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

Let $x = 0$

$z = \frac{y}{\sqrt{3}}$

Let $y = 1$



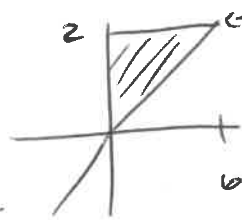
$\tan\phi = \frac{1}{\frac{1}{\sqrt{3}}} = \sqrt{3}$

$\phi = 60°$ or $\frac{\pi}{3}$

3. (7 points) Evaluate the integral $\int_{x=0}^{6} \int_{y=x/3}^{2} \frac{\sin(y)}{y}\, dy dx$ by reversing the order of integration.

$\frac{x}{3} \le y \le 2$

$0 \le x \le 6$



$0 \le y \le 2$

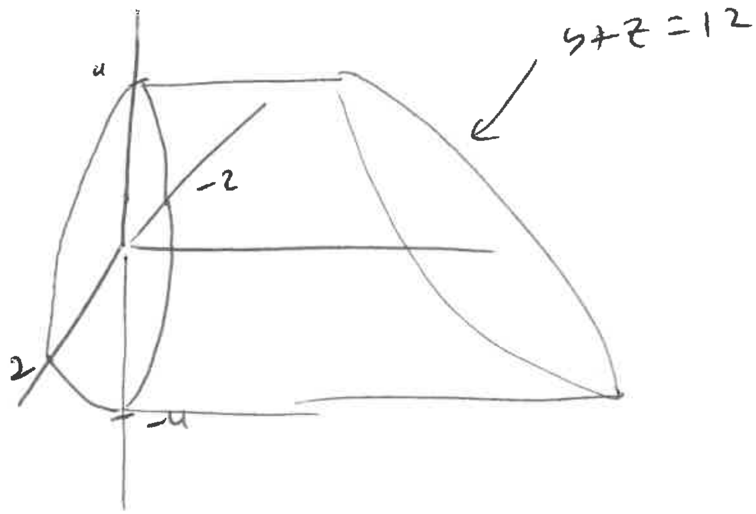
$0 \le x \le 3y$

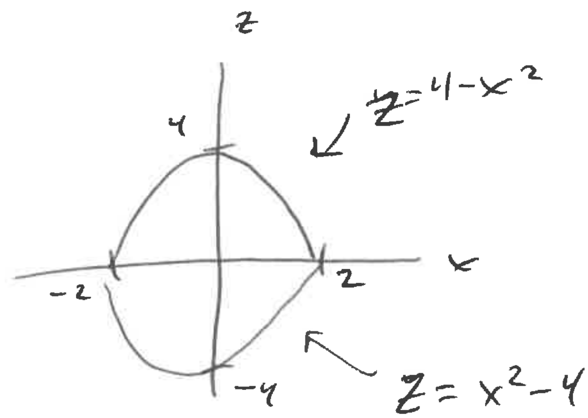
$$\int_{y=0}^{2} \int_{x=0}^{3y} \frac{\sin(y)}{y}\, dx\, dy = \int_{y=0}^{2} x \frac{\sin(y)}{y} \Big|_0^{3y} dy = \int_{y=0}^{2} \frac{3y \sin(y)}{y}\, dy$$

$$= \int_0^2 3\sin(y)\, dy = -3\cos(y)\Big|_0^2 = -3\cos(2) - -3\cos(0)$$

$$= 3 - 3\cos(2)$$

4. (10 points) Set up the integral used to find the volume of the solid enclosed by $z = x^2 - 4$, $z = 4 - x^2$, $y = 0$, and $y + z = 12$.





Left $y=0$

Right $y=12-z$

$-2 \le x \le 2$

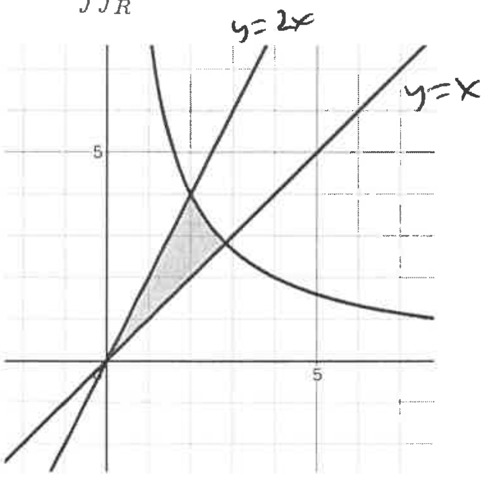
$x^2-4 \le z \le 4-x^2$

$$\int_{x=-2}^{2} \int_{z=x^2-4}^{4-x^2} \int_{y=0}^{12-z} 1 \, dy \, dz \, dx$$

5. (11 points) R is the region enclosed by the lines $y = 2x$, $y = x$, and the hyperbola $xy = 8$.

Use the given transformation to change the variables for the given integral. You only have to give the new integral. You do not need to compute the new integral.

$$\iint_R (4y)\,dA \qquad x = u \qquad y = \frac{v}{u}$$



Region R

$$J = \begin{vmatrix} 1 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{1}{u}$$

$$|J| = \frac{1}{u}$$

$\underline{y = x}$

$\frac{v}{u} = u$

$v = u^2$

$\underline{y = 2x}$

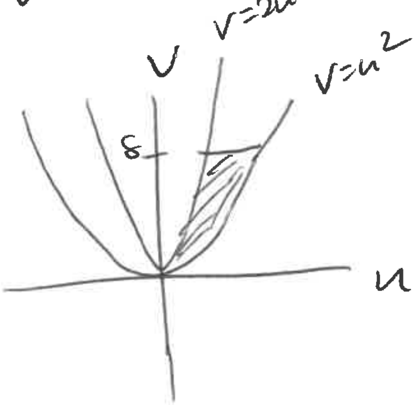
$\frac{v}{u} = 2u$

$v = 2u^2$

$\underline{xy = 8}$

$u \cdot \frac{v}{u} = 8$

$v = 8$



$0 \le v \le 8$

$\sqrt{\frac{v}{2}} \le u \le \sqrt{v}$

$$\int_{v=0}^{8} \int_{u=\sqrt{\frac{v}{2}}}^{\sqrt{v}} 4 \cdot \frac{v}{u} \cdot \frac{1}{u}\, du\, dv$$

6. (10 points) Set up the integral in **polar** that will find the surface area of the function $z = x^2 + y^3$ above the region on the xy-plane that is inside the circle $x^2 + y^2 = 4y$.

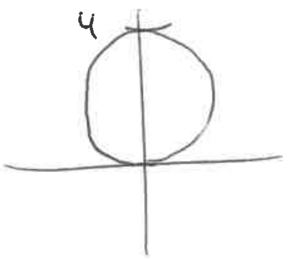
$$SA = \iint_D \sqrt{(Z_x)^2 + (Z_y)^2 + 1}\, dA = \iint_D \sqrt{(2x)^2 + (3y^2)^2 + 1}\, dA$$

$$= \iint_D \sqrt{4x^2 + 9y^4 + 1}\, dA$$

$x^2 + y^2 = 4y$

$r^2 = 4r\sin\theta$

$r = 4\sin\theta$



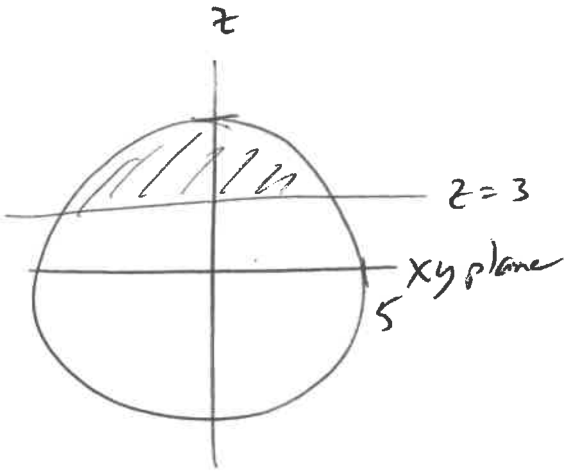
$$= \int_{\theta=0}^{\pi} \int_{r=0}^{4\sin\theta} \sqrt{4(r\cos\theta)^2 + 9(r\sin\theta)^4 + 1}\; r\,dr\,d\theta$$

$0 \le \theta \le \pi$

$0 \le r \le 4\sin\theta$

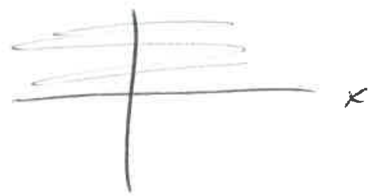
7. (10 points) The solid E is the region that is within the sphere $x^2 + y^2 + z^2 = 25$ and above the plane $z = 3$.
The solid E is also in the half-space $y \geq 0$.

<u>Set up</u> the integral that would find the volume of the solid E.



and $y \geq 0$

Region D



$z = 3$

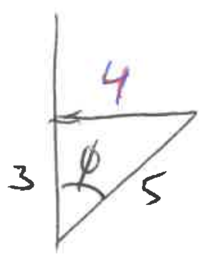
$\rho \cos\phi = 3$

$\rho = 3\sec\phi$

$0 \leq \theta \leq \pi$

$3\sec\phi \leq \rho \leq 5$

$0 \leq \phi \leq \arccos\left(\frac{3}{5}\right)$



$\cos\phi = \frac{3}{5}$

$\phi = \arccos\left(\frac{3}{5}\right)$

$V = \iiint_E 1 \, dV$ 1.5

1.5

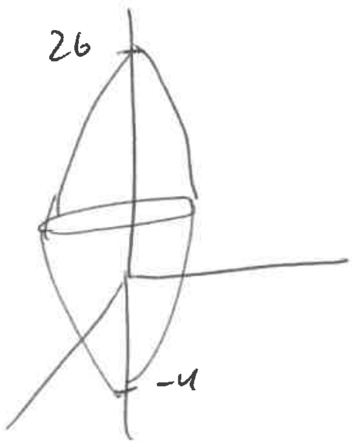
$$= \int_{\theta=0}^{\pi} \int_{\phi=0}^{\arccos(3/5)} \int_{\rho=3\sec\phi}^{5} 1 \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

arctan(4/3)

1.5 1 1 1 1 1.5

8. (10 points) The solid E is the region that is bounded by $z = 3x^2 + 3y^2 - 4$ and $z = 26 - 2x^2 - 2y^2$ and has $x \geq 0$.

Set up the integral $\iiint_E y \, dV$.



Top $z = 26 - 2x^2 - 2y^2 = 26 - 2r^2$

Bottom $z = 3x^2 + 3y^2 - 4 = 3r^2 - 4$

Intersection

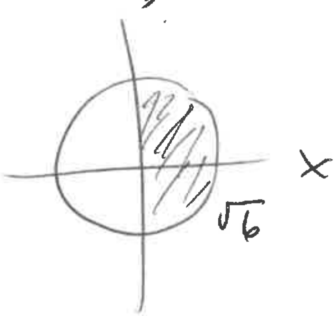
$3x^2 + 3y^2 - 4 = 26 - 2x^2 - 2y^2$

$5x^2 + 5y^2 = 30$

$x^2 + y^2 = 6$

Region D

also $x \geq 0$



$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$0 \leq r \leq \sqrt{6}$

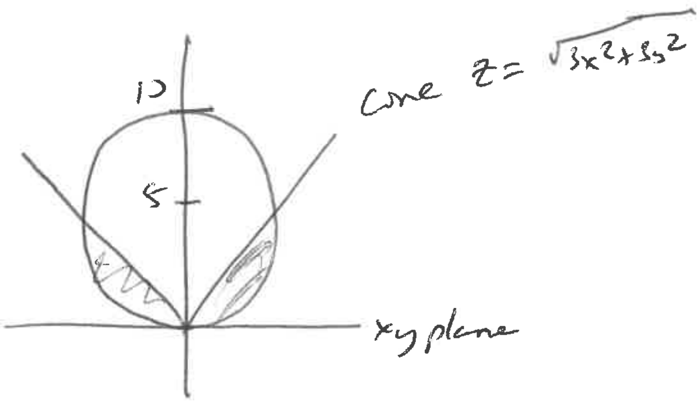
$$\int_{\theta=-\frac{\pi}{2}}^{\pi/2} \int_{r=0}^{\sqrt{6}} \int_{z=3r^2-4}^{26-2r^2} r\sin\theta \cdot r \, dz \, dr \, d\theta$$

9. (11 points) The solid E is the region that is within the sphere $x^2 + y^2 + z^2 = 10z$ and below the cone $z = \sqrt{3x^2 + 3y^2}$.
The solid E is also in the half-space $y \geq 0$.

Set up the integral $\iiint_E x \, dV$.

$\hookrightarrow \rho^2 = 10\rho\cos\phi$

$\rho = 10\cos\phi$



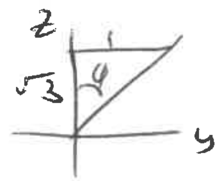
$z = \sqrt{3x^2+3y^2}$

let $x=0$

$z = \sqrt{3y^2} = \sqrt{3}\, y$

Let $y=1$

$z = \sqrt{3}$



$\tan\phi = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\phi = \frac{\pi}{6}$

$0 \leq \theta \leq \pi$

$0 \leq \rho \leq 10\cos\phi$

$\frac{\pi}{6} \leq \phi \leq \frac{\pi}{2}$

$$\int_{\theta=0}^{\pi} \int_{\phi=\frac{\pi}{6}}^{\pi/2} \int_{\rho=0}^{10\cos\phi} \rho\sin\phi\cos\theta \; \rho^2\sin\phi \; d\rho \, d\phi \, d\theta$$

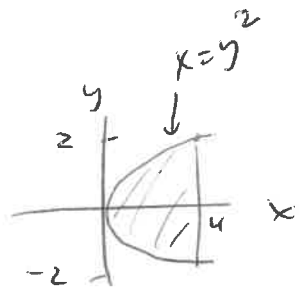
10. (10 points) Rewrite the integral in the order of **dx dz dy**.

$$\int_{x=0}^{4}\int_{y=-\sqrt{x}}^{\sqrt{x}}\int_{z=0}^{12-3x} f(x,y,z)\ dz\ dy\ dx$$

Region D

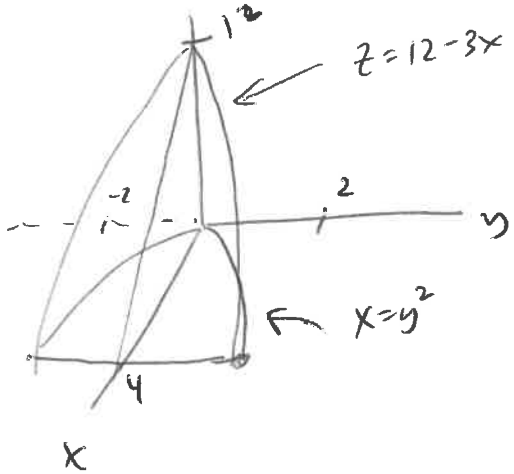
$-\sqrt{x} \le y \le \sqrt{x}$

$0 \le x \le 4$

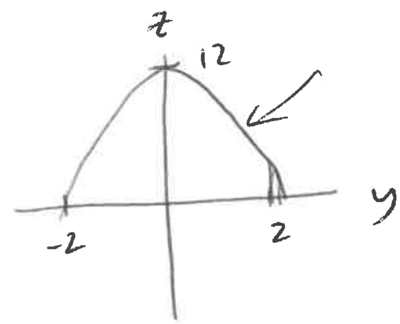


Top $z = 12-3x$

Bottom $z = 0$



project on yz plane



$z = 12-3x$

$z = 12-3y^2$

dz dy ↑

$0 \le z \le 12-3y^2$

$-2 \le y \le 2$

front $x = \dfrac{12-z}{3}$

Back $x = y^2$

$$\int_{y=-2}^{2}\int_{z=0}^{12-3y^2}\int_{x=y^2}^{\frac{12-z}{3}} f(x,y,z)\ dx\ dz\ dy$$