Problem 1

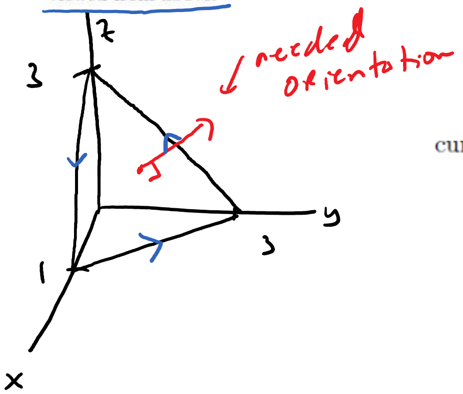
Stokes Thrm says

These are equivalent

$$\int_C F \cdot dr = \iint_S \text{curl} F \, dS$$

$P \quad Q \quad R$

Let $\mathbf{F} = \langle xz,\ 2xy,\ 3xy \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the boundary of the part of the plane $3x + y + z = 3$ in the first octant and C is oriented counterclockwise as viewed from above.



$$\text{curl}\,\mathbf{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$$

$$= \langle 3x - 0,\ x - 3y,\ 2y - 0 \rangle$$

$$= \langle 3x,\ x - 3y,\ 2y \rangle$$

Surface.

$x = x$

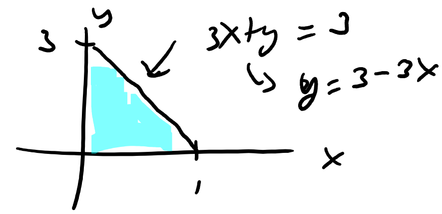
$y = y$

$z = 3 - 3x - y$

$r_x \times r_y = \langle 3, 1, 1 \rangle$

orientation ✓

Region D for surface S



$0 \le x \le 1$

$0 \le y \le 3 - 3x$

$$\int_C F \cdot dr = \iint_S \text{curl} F \cdot dS = \iint_D \langle 3x, x-3y, 2y \rangle \cdot \langle 3, 1, 1 \rangle \, dA$$

$$= \int_{x=0}^{1} \int_{y=0}^{3-3x} 9x + x - 3y + 2y \, dy\, dx = \int_{x=0}^{1} \int_{y=0}^{3-3x} 10x - y \, dy\, dx$$

$$= \int_{x=0}^{1} 10xy - \frac{y^2}{2} \Big|_0^{3-3x} dx = \int_{x=0}^{1} 10x(3-3x) - \frac{1}{2}(3-3x)^2 \, dx$$

$$= \int_{x=0}^{1} 30x - 30x^2 - \frac{1}{2}(3-3x)^2 \, dx$$

$$\int \frac{1}{2}(3-3x)^2 \, dx$$

$u = 3 - 3x$

$du = -3\,dx$

$$= \int \frac{1}{2}\left(\frac{-1}{3}\right) u^2 \, du$$

$$\int_{x=0} 30x - 30x^2 - \frac{1}{2}(3 \ldots$$

$$= 15x^2 - 10x^3 - \frac{1}{2}\cdot\left(\frac{-1}{3}\right)\left(\frac{1}{3}\right)(3-3x)^3 \Bigg|_{x=0}^{1}$$

$u = 3-3x$

$du = -3\,dx$

$-\frac{1}{3}\,du = dx$

$\int \frac{1}{2}(3 \ldots$

$= \frac{1}{2}\left(\frac{-1}{3}\right)\frac{1}{3}u^3$

$= \frac{1}{2}\left(\frac{-1}{3}\right)\left(\frac{1}{3}\right)(3-3x)^3$

$$= 15 - 10 + \frac{1}{18}(0)^3 - \left[0 - 0 + \frac{1}{18}(3)^3\right]$$

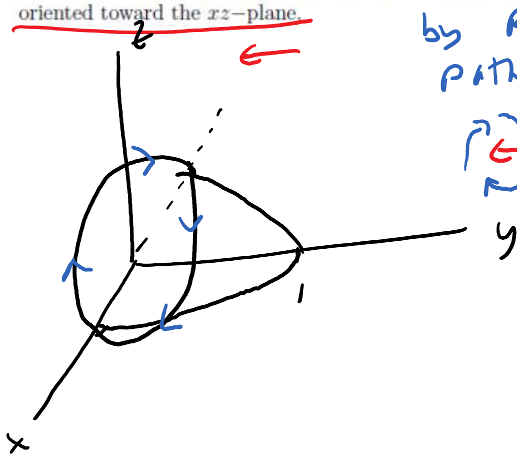
$$= 5 - \frac{1}{18}\cdot 27 = 5 - \frac{1}{2}\cdot 3 = 5 - \frac{3}{2} = \frac{7}{2}$$

Problem 2

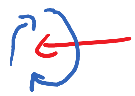
Use Stokes's Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$

$\mathbf{F} = \langle yz^3,\ \sin(xyz),\ x \rangle$

S is the part of the paraboloid $y = 1 - x^2 - z^2$ that lies to the right of the xz−plane, oriented toward the xz−plane.



by Right hand Rule.
path orientation is

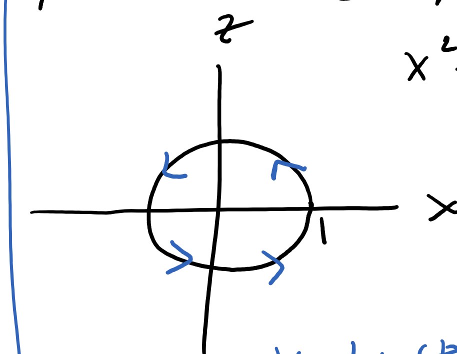


surface

Stokes Thrm says
These are equivalent

$$\int_C F \cdot dr = \iint_S \text{curl } F \, dS$$

path on xz plane.



$x^2 + z^2 = 1$

$(y = 0)$

$x = 1 \cos\theta$

$z = 1 \sin\theta$

orientation ✓

$$r(\theta) = \langle \cos\theta,\ 0,\ \sin\theta \rangle$$

$$r'(\theta) = \langle -\sin\theta,\ 0,\ \cos\theta \rangle$$

$$F = \langle 0,\ \sin(0),\ \cos\theta \rangle$$

$$F = \langle 0,\ 0,\ \cos\theta \rangle$$

$$F \cdot dr = \cos^2\theta$$

$$\iint_S \text{curl } F \cdot dS = \int_C F \cdot dr = \int_0^{2\pi} \cos^2\theta \, d\theta = \int_0^{2\pi} \frac{1}{2}(1 + \cos 2\theta) \, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi} = \frac{1}{2}\left(2\pi + \frac{1}{2}\sin(4\theta)\right) - 0$$

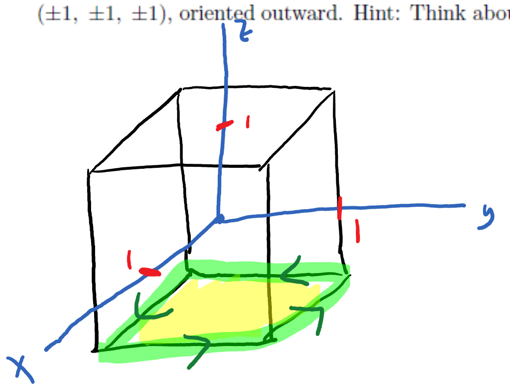
$$= \frac{1}{2}(2\pi + 0) = \pi$$

Problem 3

Use Stokes's Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$

$\mathbf{F} = \langle xyz,\ xy,\ x^2yz \rangle$

S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm 1,\ \pm 1,\ \pm 1)$, oriented outward. Hint: Think about the last example in the lecture.



S is the surface of the box oriented outward.

C is the path which is the base of each side. Orientation is counter clockwise as seen from above.

S_2 is the yellow side of the cube. The orientation of this surface is upward.

Thus.

$$\underbrace{\iint_S \text{curl} F \cdot dS = \overbrace{\int_C F \cdot dr}_{}}_{\text{Stokes thrm}} \overbrace{= \iint_{S_2} \text{curl} F \cdot dS_2}^{\text{Stokes thrm.}}$$

S_2 | $x = x$, $y = y$, $z = -1$

$r(x,y) = \langle x, y, -1 \rangle$

$-1 \le x \le 1$

$-1 \le y \le 1$

$r_x \times r_y = \langle 0, 0, 1 \rangle \leftarrow$ upward orientation ✓

$\text{Curl} F = \langle x^2 z - 0,\ xy - 2xyz,\ y - xz \rangle$

$$\text{curl}\,F \cdot r_x \times r_y = y - xz = y - x(-1) = y + x$$

$$\iint_{S_2} \text{curl}\,F \cdot dS = \iint_D y + x \, dA = \int_{x=-1}^{1} \int_{y=-1}^{1} y + x \, dy \, dx$$

$$= \int_{x=-1}^{1} \frac{y^2}{2} + xy \Big|_{-1}^{1} dx = \int_{x=-1}^{1} \frac{1}{2} + x - \left(\frac{1}{2} - x\right) dx$$

$$= \int_{x=-1}^{1} \frac{1}{2} + x - \frac{1}{2} + x \, dx = \int_{x=-1}^{1} 2x\,dx = x^2 \Big|_{-1}^{1}$$

$$= 1^2 - (-1)^2 = 1 - 1 = 0$$