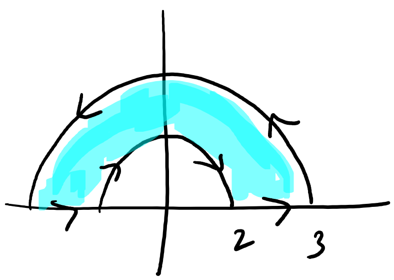
Problem 1

Evaluate $\oint_{\partial D} y^2dx + 3xydy$, where D is the region in the upper half-plane between the circles $x^2+y^2=4$ and $x^2+y^2=9$. Assume the region is bounded by a positively oriented curve.

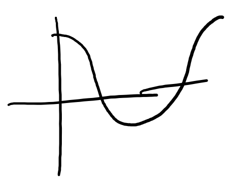
Region D



polar region

$2 \le r \le 3$

$0 \le \theta \le \pi$



$P = y^2 \qquad P_y = 2y$

$Q = 3xy \qquad Q_x = 3y$

$$\oint_{\partial D} y^2 dx + 3xy\,dy = \iint_D 3y - 2y \, dA$$

$$= \iint_D y \, dA = \int_{\theta=0}^{\pi} \int_{r=2}^{3} r\sin\theta \cdot r \, dr \, d\theta$$

$$= \int_0^{\pi} \sin\theta \, d\theta \cdot \int_2^3 r^2 \, dr$$

$$= -\cos\theta \Big|_0^{\pi} \cdot \frac{1}{3} r^3 \Big|_2^3$$

$$= \left[-\cos(\pi) - (-\cos 0)\right] \cdot \left[9 - \frac{8}{3}\right]$$

$$= (1 + 1)\left(\frac{19}{3}\right) = \frac{38}{3}$$

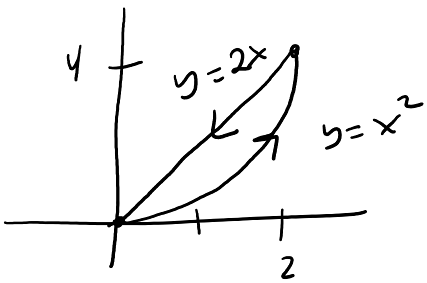
Problem 2

Evaluate the line integral shown below where C is the path From the point $(0,0)$ to the point $(2,4)$ along the function $y = x^2$ and then from the point $(2,4)$ back to the point $(0,0)$ along the path $y = 2x$

$$\int_C 5xy\,dx + x^3\,dy$$

$P = 5xy \qquad P_y = 5x$

$Q = x^3 \qquad Q_x = 3x^2$



$$\int_C 5xy\,dx + x^3\,dy = \iint_D 3x^2 - 5x \; dA$$

$$= \int_{x=0}^{2} \int_{y=x^2}^{2x} 3x^2 - 5x \; dy\,dx = \int_{x=0}^{2} (3x^2 - 5x)\,y \Big|_{y=x^2}^{2x} dx$$

$$= \int_{x=0}^{2} (3x^2 - 5x)(2x - x^2)\,dx$$

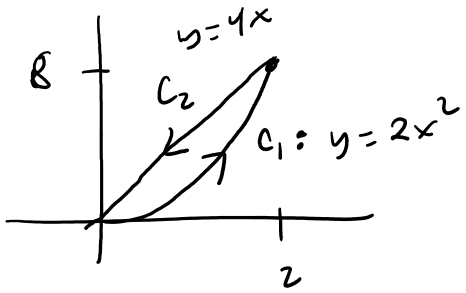
$$= \int_{x=0}^{2} 6x^3 - 10x^2 - 3x^4 + 5x^3\,dx$$

$$= \int_{x=0}^{2} 11x^3 - 10x^2 - 3x^4\,dx = \left(\frac{11x^4}{4} - \frac{10x^3}{3} - \frac{3x^5}{5} \right) \Bigg|_0^2$$

$$= 44 - \frac{80}{3} - \frac{96}{5} = \frac{-28}{15}$$

Problem 3

Use Green's Theorem to find the area bounded between $y = 2x^2$ and $y = 4x$. Assume that there is a positive orientation.



Intersection point

$2x^2 = 4x$

$2x^2 - 4x = 0$

$2x(x-2) = 0$

Path $C = C_1 + C_2$

we need to pick P and Q such that $Q_x - P_y = 1$

Lets try

$P = 0$

$Q = x$

$$\text{Area} = \iint_D 1 \, dA = \int_C P\,dx + Q\,dy$$

$$= \int_C x\,dy$$

we need to do both path separately.

C_1) $x = t$ $\quad y = 2t^2$ $\quad 0 \le t \le 2$

$r = \langle t, 2t^2 \rangle$

This matches the direction of the path..

$$\int_{C_1} x\,dy = \int_0^2 t \cdot 4t \, dt = \int_0^2 4t^2 dt = \frac{4t^3}{3}\Big|_0^2 = \frac{32}{3}$$

C_2) $x = t$ $\quad y = 4t$ $\quad 0 \le t \le 2$

This is Backwards than the desired path. change

$$\int_{C_2} x\,dy = \int_0^2 t \cdot 4\,dt = 2t^2 \Big|_0^2 = 8$$

the sign at the end

answer for C_2 should be -8

$$\text{Area} = \frac{32}{3} - 8 = \frac{8}{3}$$