1. Evaluate $\int_C (\sin x + \cos y)ds$, where C is the line segment going from the point $(0,0)$ to the point $(3\pi, 4\pi)$.

(a) $\frac{2}{3\pi}$

(b) $\frac{10}{3}$

(c) $-\frac{10}{3}$

(d) $30\pi^2$

(e) $-\frac{2}{3\pi}$

$\int_C f(x,y)ds \qquad C: r(t), \quad a \le t \le b$

$$\int_C f(x,y)ds = \int_a^b f(r(t))|r'(t)|dt$$

Line: $\vec{r}(t) = \vec{r_0} + t\vec{v}$

$= \langle 0,0 \rangle + t\langle 3\pi, 4\pi \rangle, \quad 0 \le t \le 1$

$\boxed{r(t) = \langle 3\pi t, 4\pi t \rangle}$ $\qquad r' = \langle 3\pi, 4\pi \rangle$

$$\int_0^1 (\sin(3\pi t) + \cos(4\pi t))\sqrt{9\pi^2 + 16\pi^2}\,dt$$

$$= \left[-\frac{1}{3\pi}\cos(3\pi t) + \cancel{\frac{1}{4\pi}\sin(4\pi t)}^{\,0} \right]\Big|_0^1 \sqrt{25\pi^2}$$

$$= 5\pi\left[-\frac{1}{3\pi}(-1) - \left(-\frac{1}{3\pi}\cos(0) \right) \right] = 5\pi \cdot \frac{2}{3\pi} = \frac{10}{3}$$

2. Find the surface area of the part of the plane $\boxed{z - x - y = 4}$ that lies inside the cylinder $x^2 + y^2 = 9$.

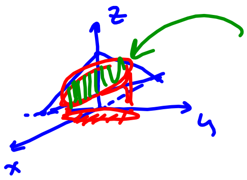
(a) $9\sqrt{2}\pi$

(b) $6\sqrt{3}\pi$

(c) $\sqrt{19}\pi$

(d) $9\sqrt{3}\pi$

(e) $2\sqrt{19}\pi$



$S(A) = \iint_D |r_u \times r_v|\, dA \qquad r(u,v) =$ surface parameterized.

$z = 4 + x + y \qquad u = x,\ v = y$

D = domain of surface

$S: r(u,v) = \langle u, v, 4+u+v \rangle,$

$u^2 + v^2 \le 9 \leftarrow$ domain D

$r_u = \langle 1, 0, 1 \rangle \qquad r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$

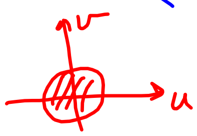
$r_v = \langle 0, 1, 1 \rangle$

$= \langle -1, -1, 1 \rangle$

$SA = \iint_D |\langle -1, -1, 1 \rangle|\, dA$

$= \iint_D \sqrt{3}\, dA$

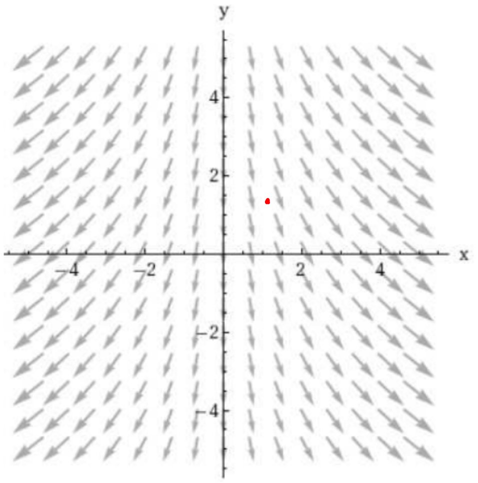
$\int_0^{2\pi}\int_0^3 \sqrt{3}\, r\, dr\, d\theta$

D: 

$0 \le \theta \le 2\pi$

$0 \le r \le 3$

3. Which of the following vector functions **F** has the vector field shown below?



(1,1)

~~(a)~~ $\mathbf{F} = \langle -2x, -8 \rangle$ $F(1,1) = \langle -2, -8 \rangle$ ↙

→ (b) $\mathbf{F} = \langle 2x, -8 \rangle$ $F(1,1) = \langle 2, -8 \rangle$ ↘

(c) $\mathbf{F} = \langle 2x, 8 \rangle$ $F(1,1) = \langle 2, 8 \rangle$ ↗

(d) $\mathbf{F} = \langle -2x, 8 \rangle$ $F(1,1) = \langle -2, 8 \rangle$ ↑

(e) None of these

4. Using the Divergence Theorem, find the flux of $\mathbf{F} = \langle e^z + \cos y,\ y^3 + \sin(xz),\ z^3 + 2e^{-x} \rangle$ across the positively oriented surface S, where $\boxed{S}$ is the part of the cylinder $y^2 + z^2 = 1$ that lies between the planes $x = 0$ and $x = 3$.

(a) $\frac{3\pi}{2}$

(b) 6π

(c) $\frac{9\pi}{2}$

(d) 3π

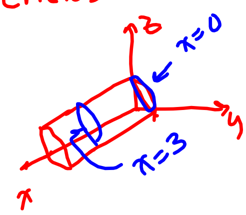
(e) $\frac{\pi}{2}$

$$\text{Flux } F = \iint_S F \cdot dS$$

Div Theorem: $\iint_S F \cdot dS = \iiint_E \operatorname{div} F \, dV$

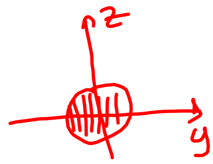
V = volume S encloses

$$\iiint_{\boxed{E}} (0 + 3y^2 + 3z^2)\, dV$$



$$\iint \left[\int_{x=0}^{x=3} (3y^2 + 3z^2)\, dx \right] dA$$

D = yz plane



$$= \int_0^{2\pi} \int_0^1 \int_{x=0}^{x=3} 3r^2 \, dx \, r\, dr\, d\theta$$

$0 \le \theta \le 2\pi$

$0 \le r \le 1$

$$= \int_0^{2\pi} d\theta \int_0^1 3r^3\, dr \int_0^3 dx$$

$$= (2\pi)\left(\frac{3r^4}{4} \right)\Big|_0^1 \cdot 3 = \frac{9\pi}{2}$$

5. Find the work done by the force field $\mathbf{F} = \langle \overset{P}{x^2 + y^2}, \overset{Q}{xy} \rangle$ in moving a particle along the curve C: $\mathbf{r}(t) = \langle t^2, t^3 \rangle$, $0 \le t \le 1$.

(a) $\frac{20}{21}$

(b) $\frac{52}{21}$

(c) $\frac{23}{24}$

(d) $\frac{22}{56}$

(e) $\frac{167}{68}$

$W = \int F \cdot dr$ $\qquad \frac{\partial Q}{\partial x} = y \qquad \frac{\partial P}{\partial y} = 2y$

$$W = \int_0^1 F(r(t)) \cdot r'(t)\,dt$$

$$= \int_0^1 \langle t^4 + t^6, t^5 \rangle \cdot \langle 2t, 3t^2 \rangle\,dt$$

$$= \int_0^1 \left(2t^5 + 2t^7 + 3t^7\right) dt$$

6. Find $\oint_C y^2 dx + x dy$ where C is the triangle with vertices $(0,0)$, $(1,1)$ and $(0,1)$. Assume positive (counterclockwise) orientation.

(a) $\frac{7}{6}$

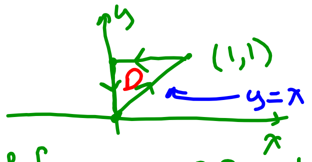
(b) $\frac{1}{6}$

(c) $-\frac{7}{6}$

(d) 0

(e) $-\frac{1}{6}$

$\int_C P dx + Q dy$ $\qquad$ is C closed?



$0 \le x \le y$

$0 \le y \le 1$

$$\oint_C y^2 dx + x\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

$$= \iint_D (1 - 2y)\,dA$$

$$= \int_0^1 \int_0^y (1-2y)\,dx\,dy$$

$$\int_0^1 (1-2y) \cdot x \Big|_{x=0}^{x=y}$$

$$\int_0^1 (1-2y)(y)$$

$$\boxed{\int_0^1 \int_x^1 (1-2y)\,dy\,dx}$$

7. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle 2xy^3z^4,\ 3x^2y^2z^4,\ 4x^2y^3z^3 \rangle$ and C is any path from the point $(0,0,0)$ to the point $(1,1,1)$.

(a) 2
(b) 1
(c) -2
(d) 0
(e) -1

F must be conservative since it is path independent.

$\int 2xy^3z^4\,dx = x^2y^3z^4 + g(y,z)$

$\int 3x^2y^2z^4\,dy = x^2y^3z^4 + h(x,z)$

$\int 4x^2y^3z^3\,dz = x^2y^3z^4 + k(x,y)$

$f(x,y,z) = x^2y^3z^4$ $\qquad \int_C F\cdot dr = \int_C \nabla f \cdot dr$

$= f(1,1,1) - f(0,0,0) = \boxed{1}$

8. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle e^{x-1},\ xy \rangle$ and C: $\mathbf{r}(t) = \langle t^2,\ t^3 \rangle,\ \ 0 \le t \le 1$.

(a) $\frac{11}{22} - \frac{1}{2e}$
(b) $\frac{17}{22}$
(c) $\frac{11}{8}$
(d) $\frac{11}{8} - \frac{e}{2}$
(e) $\frac{11}{8} - \frac{1}{e}$

$\int_C F\cdot dr = \int_0^1 F(r(t))\cdot r'(t)\,dt$

$= \int_0^1 \langle e^{t^2-1}, t^5 \rangle \cdot \langle 2t, 3t^2 \rangle\,dt$

$= \int_0^1 \left(2te^{t^2-1} + 3t^7\right) dt$

$u = t^2 - 1$
$du = 2t\,dt$

$= \left(e^{t^2-1} + \frac{3t^8}{8}\right)\Big|_0^1$

$= 1 + \frac{3}{8} - e^{-1} = \frac{11}{8} - \frac{1}{e}$

9. Which of the following integrals is the correct set up in order to evaluate $\iint_S xz\,dS$ where S is the part of the sphere $x^2+y^2+z^2=1$ that lies between the planes $z=0$ and $z=\frac{1}{2}$? Note: If we parameterize the sphere $x^2+y^2+z^2=\rho^2$ by $\mathbf{r}(\theta,\phi)=\langle \rho\sin(\phi)\cos(\theta),\ \rho\sin(\phi)\sin(\theta),\ \rho\cos(\phi)\rangle$, then $|\mathbf{r}_\theta\times\mathbf{r}_\phi|=\rho^2\sin(\phi)$.

(a) $\int_0^{2\pi}\int_{\pi/3}^{\pi/2} \sin^2(\phi)\cos(\phi)\cos(\theta)\,d\phi d\theta$

(b) $\int_0^{2\pi}\int_{\pi/3}^{\pi/2} \sin^3(\phi)\cos(\phi)\cos(\theta)\,d\phi d\theta$

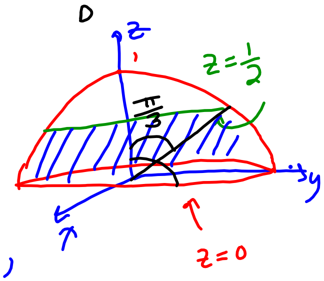
(c) $\int_0^{2\pi}\int_0^{\pi/3} \sin^3(\phi)\cos(\phi)\cos(\theta)\,d\phi d\theta$

(d) $\int_0^{2\pi}\int_0^{\pi/3} \sin^2(\phi)\cos^2(\phi)\,d\phi d\theta$

(e) $\int_0^{2\pi}\int_{\pi/6}^{\pi/2} \sin^2(\phi)\cos(\phi)\cos(\theta)\,d\phi d\theta$

$$\iint_S f(x,y,z)\,dS = \iint_D f(r(u,v))\,|r_u\times r_v|\,dA$$

$\boxed{\text{Since } \rho=1}$



$r(\theta,\phi)=\langle \sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi\rangle,$

$\frac{\pi}{3}\le\phi\le\frac{\pi}{2}$

$0\le\theta\le 2\pi$

$z=\rho\cos\phi$

$z=\cos\phi$

$z=\frac{1}{2}:\ \frac{1}{2}=\cos\phi$

$\phi=\frac{\pi}{3}$

$$\int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sin\phi\cos\theta\cdot\cos\phi)(\sin\phi)\,d\phi\,d\theta$$

10. Which of the following integrals is a result of using Stokes' Theorem to find $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}=\langle -y^2, x, z^2\rangle$ and C is the curve of intersection of $z=9-x^2-y^2$ and the xy plane. Orient C to be counterclockwise when looking from above.

(a) $\int_0^{2\pi}\int_0^3 (r+2r^2\sin\theta)\,drd\theta$

(b) $\int_0^{\pi}\int_0^3 (1+2r\sin\theta)\,drd\theta$

(c) $\int_0^{2\pi}\int_0^3 (r-2r^2\sin\theta)\,drd\theta$

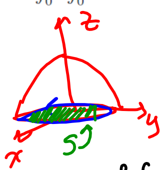
(d) $\int_0^{2\pi}\int_0^3 (1+2r\sin\theta)\,drd\theta$

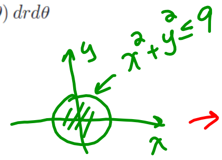
(e) $\int_0^{\pi}\int_0^3 (1-2r\sin\theta)\,drd\theta$

$$\int_C F\cdot ds = \iint_S \operatorname{curl} F\cdot dS$$

S = surface the curve encloses

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{vmatrix} = \langle 0,0,1+2y\rangle$$





$\rightarrow R(r,\theta)=\langle r\cos\theta,\ r\sin\theta,\ 0\rangle,$ $0\le r\le 3$, $0\le\theta\le 2\pi$

$$\iint_S \operatorname{curl} F\cdot dS = \iint_D \operatorname{curl} F(R(r,\theta))\cdot(R_r\times R_\theta)\,dA$$

$$= \int_0^{2\pi}\int_0^3 \langle 0,0,1+2r\sin\theta\rangle\cdot\langle 0,0,r\rangle\,dr\,d\theta$$

$$R_r\times R_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle 0,0,r\cos^2\theta + r\sin^2\theta\rangle = \langle 0,0,r\rangle$$

$$\int_0^{2\pi}\int_0^3 (r+2r^2\sin\theta)\,dr\,d\theta$$

11. Given $\mathbf{F}=\langle x+y^2,\ 2xy+y^2\rangle$, find $f(-1,2)$ where f is the potential function of $\mathbf{F}$.

(a) $-\frac{2}{3}$

(b) $-\frac{5}{6}$

(c) $\frac{2}{3}$

(d) 2

(e) $\frac{23}{6}$

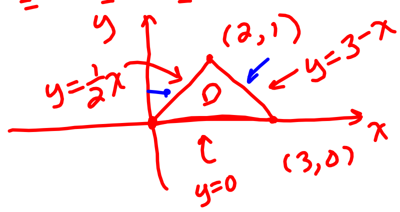
$$f(x,y) = xy^2 + \frac{x^2}{2} + \frac{y^3}{3}$$

12. (7 pts) Let f be a scalar field and let $\mathbf{F}$ be a vector field. Determine whether each expression is meaningful or meaningless (circle one). **If the expression is meaningful, circle whether the expression is a vector or a scalar.**

a.) curl $\mathbf{F}$	meaningful (vector or scalar)	meaningless
b.) grad$f$	meaningful (vector or scalar)	meaningless
c.) grad (div $\mathbf{F}$)	meaningful (vector or scalar)	meaningless
d.) (grad f) $\times$ (curl $\mathbf{F}$)	meaningful (vector or scalar)	meaningless
d.) curl f	meaningful (vector or scalar)	meaningless
e.) grad $\mathbf{F}$	meaningful (vector or scalar)	meaningless
f.) div (grad$f$)	meaningful (vector or scalar)	meaningless

f

13. (10 pts) Consider $\int_C (\sin(xy)dx + (x + \ln(y+1)dy$, where C is the triangle with vertices $(0,0)$, $(2,1)$ and $(3,0)$. Using Green's Theorem, set up the resulting double integral. Do not evaluate the integral.

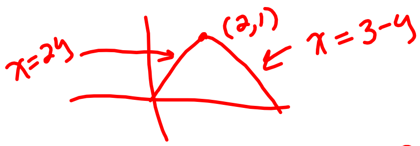


$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$\iint_D \left(1 - \cos(xy)\cdot x\right) dA$$

Type I: $0 \le y \le \frac{1}{2}x$, $0 \le x \le 2$

$0 \le y \le 3-x$, $2 \le x \le 3$



$$\int_0^1 \int_{2y}^{3-y} \left(1 - x\cos(xy)\right) dx\, dy$$

Type II: $2y \le x \le 3-y$

$0 \le y \le 1$

14. (10 pts) Set up but do not evauate the surface integral obtained by using Stokes' Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle y^2,\ z^2,\ x^2\rangle$ and where C is the curve of intersection of of the plane $x + z = 1$ and the cylinder $x^2 + y^2 = 4$, oriented counterclockwise when viewed from above.

$$\int_C F\cdot dr = \iint_S \operatorname{curl} F \cdot dS$$

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & z^2 & x^2 \end{vmatrix}$$

$S:\ z = 1 - x$

$r(u,v) = \langle u, v, 1-u\rangle \longrightarrow \boxed{\operatorname{curl} F = \langle -2z, -2x, -2y\rangle}$

$u^2 + v^2 \le 4$

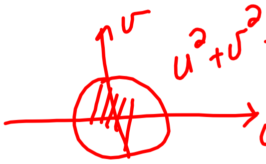
$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \langle 1, 0, 1\rangle$$

$$\iint_S \operatorname{curl} F\cdot dS = \iint_D \operatorname{curl} F(r(u,v))\cdot(r_u\times r_v)\,dA$$

$$= \iint_D \langle -2(1-u), -2u, -2v\rangle\cdot\langle 1,0,1\rangle\,dA$$

$$= \iint_D (-2 + 2u - 2v)\,dA$$

$$= \int_0^{2\pi}\int_0^2 (-2 + 2r\cos\theta - 2r\sin\theta)\,r\,dr\,d\theta$$



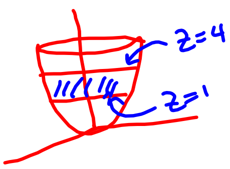
$u^2+v^2\le 4$

$0 \le \theta \le 2\pi$

$0 \le r \le 2$

15. (8 pts) Set up but do not evaluate $\iint_S xyz\,dS$ where S is the part of the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 4$.

$$\iint_S f(x,y,z)\,dS = \iint_D f(r(u,v))(|r_u\times r_v|)\,dA$$



$z = x^2 + y^2$

$z=1:\ x^2+y^2=1$

$z=4:\ x^2+y^2=4$

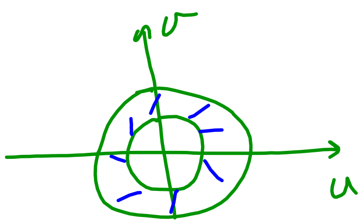
$r(u,v) = \langle u, v, u^2+v^2\rangle$

$$r_u\times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix}$$

$r_u\times r_v = \langle -2u, -2v, 1\rangle$

$$\iint_D u\cdot v\cdot(u^2+v^2)\sqrt{4u^2+4v^2+1}\,dA$$

$$\int_0^{2\pi}\int_1^2 r\cos\theta\, r\sin\theta\, r^2\sqrt{4r^2+1}\; r\,dr\,d\theta$$



$1 \le r \le 2$

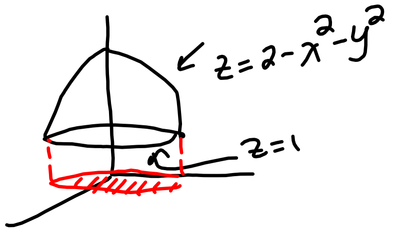
$0 \le \theta \le 2\pi$

16. (10 pts) Using the The Divergence Theorem, find the flux of $\mathbf{F} = \langle z\arctan(y^2),\ z^3\ln(x^2+1),\ z\rangle$ where S is the part of the paraboloid $x^2+y^2+z=2$ that lies above the plane $z=1$. Simplify your answer.

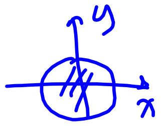
$$\text{flux } \underline{F} = \iint_S \underline{F}\cdot d\underline{S} = \iiint_E \operatorname{div} \underline{F}\, dV$$

$$1 \le z \le 2 - x^2 - y^2$$

$$1 \le z \le 2 - r^2$$



$$= \iiint_E 1\, dV$$



$$0 \le \theta \le 2\pi$$

$$0 \le r \le 1$$

these surfaces intersect when

$$1 = 2 - x^2 - y^2$$

$$x^2 + y^2 = 1$$

$$\int_0^{2\pi}\int_0^1\int_1^{2-r^2} dz\, r\, dr\, d\theta \qquad dV \to dz\, r\, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^1 r z \Big|_{z=1}^{z=2-r^2} dr\, d\theta$$

$$\int_0^{2\pi}\int_0^1 r(\underbrace{2-r^2-1}_{1-r^2})\, dr\, d\theta$$

$$\left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 (r - r^3)\, dr\right)$$

$$2\pi\left(\frac{1}{2} - \frac{1}{4}\right) = 2\pi\left(\frac{1}{4}\right) = \boxed{\frac{\pi}{2}}$$