5) The series $\sum a_n$ is defined recursively by

$$a_1 = 3 \qquad a_{n+1} = \frac{n}{n+1} a_n \text{ for } n \geq 1.$$

Determine if the series converges or diverges.

Since this is a series defines recursively, lets try the ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{n}{n+1} \cdot a_n}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = 1$$

Since this limit is 1, the ratio test fails to give any information. Now try something else.

Lets look at the individual terms of the series. To do this lets re-write the recursive formula in a form that is easier to work with.

Let $j = n+1$ Then $n = j-1$

We get

$$a_j = \frac{j-1}{j} a_{j-1} \quad \text{with } j \geq 2 \text{ and } a_1 = 3$$

now

$$a_1 = 3$$
$$a_2 = \frac{1}{2} a_1 = \frac{1}{2} \cdot 3 = \frac{3}{2}$$
$$a_3 = \frac{2}{3} a_2 = \frac{2}{3} \cdot \frac{3}{2} = \frac{3}{3}$$
$$a_4 = \frac{3}{4} a_3 = \frac{3}{4} \cdot \frac{3}{3} = \frac{3}{4}$$
$$a_5 = \frac{4}{5} a_4 = \frac{4}{5} \cdot \frac{3}{4} = \frac{3}{5}$$
$$a_6 = \frac{5}{6} a_5 = \frac{5}{6} \cdot \frac{3}{5} = 3/6$$

While this is not a formal proof, it can be shown that the general terms of the series are of the form $\frac{3}{n}$

Thus $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3}{n}$ and this is a divergent series.