Section 7.3:Additional Problems Solutions

For all of these problems we try to match up one of these trig identities for the substitution.

$\cos^2\theta = 1 - \sin^2\theta$ $\qquad$ $\sec^2\theta = 1 + \tan^2\theta$ $\qquad$ $\tan^2\theta = \sec^2\theta - 1$

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1. The expression under the square root matches the form of the identity, $\sec^2\theta = 1+\tan^2\theta$, except the constant is a 25. So multiply the identity by 25 to fix the constant.

   $25\sec^2\theta = 25 + 25\tan^2\theta$

   we need $x^2 = 25\tan^2\theta$ so let $x = 5\tan\theta$ and then $dx = 5\sec^2\theta\ d\theta$

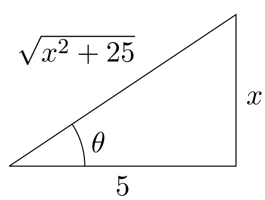
   $$\int \frac{x^3}{\sqrt{x^2+25}}\,dx = \int \frac{(5\tan\theta)^3}{\sqrt{25\tan^2\theta+25}} 5\sec^2\ d\theta = \int \frac{5^4\tan^3\theta\sec^2\theta}{\sqrt{25\sec^2\theta}}\,d\theta = \int \frac{5^4\tan^3\theta\sec^2\theta}{5\sec\theta}\,d\theta$$

   $$= \int 5^3\tan^3\theta\sec\theta\ d\theta = 125\int \tan^2\theta\tan\theta\sec\theta\ d\theta$$

   Now we want to use the $u-$sub of $u = \sec\theta$ with $du = \tan\theta\sec\theta\ d\theta$ so use the identity $\tan^2\theta = \sec^2\theta - 1$ to get.

   $$= 125\int(\sec^2\theta - 1)\theta\tan\theta\sec\theta\ d\theta = 125\int(u^2-1)du = 125\left(\frac{u^3}{3} - u\right) + C$$

   $$= 125\left(\frac{1}{3}\sec^3\theta - \sec\theta\right) + C$$

   From the triangle we see that $\sec\theta = \dfrac{\sqrt{x^2+25}}{5}$.

   Answer: $= \dfrac{125}{3}\left(\dfrac{\sqrt{x^2+25}}{5}\right)^3 - \dfrac{125\sqrt{x^2+25}}{5} + C$ or $\dfrac{1}{3}\left(\sqrt{x^2+25}\right)^3 - 25\sqrt{x^2+25} + C$

2. The expression under the square root matches the form of the identity, $\sec^2\theta = 1+\tan^2\theta$, except the constant is a 25. So multiply the identity by 16 to fix the constant.

   $16\sec^2\theta = 16 + 16\tan^2\theta$

   we need $25x^2 = 16\tan^2\theta$ so let $5x = 4\tan\theta$ or $x = \dfrac{4}{5}\tan\theta$ and then $dx = \dfrac{4}{5}\sec^2\theta\ d\theta$

   $$\int \frac{1}{x^2\sqrt{25x^2+16}}\,dx = \int \frac{1}{\left(\frac{4}{5}\tan\theta\right)^2\sqrt{16\tan^2\theta+16}}\frac{4}{5}\sec^2\theta\ d\theta = \int \frac{\frac{4}{5}\sec^2\theta}{\left(\frac{4}{5}\right)^2\tan^2\theta\sqrt{16\sec^2\theta}}\,d\theta$$

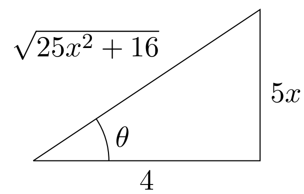
   $$= \int \frac{\sec^2\theta}{\frac{4}{5}\tan^2\theta\cdot 4\sec\theta}\,d\theta = \int \frac{\sec\theta}{\frac{16}{5}\tan^2\theta}\,d\theta = \int \frac{5}{16}\frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\,d\theta = \int \frac{5}{16}\frac{\cos\theta}{\sin^2\theta}\,d\theta$$

Now we want to use the u-sub of $u = \sin\theta$ with $du = \cos\theta\ d\theta$. This will give the integral

$$= \int \frac{5}{16}\frac{1}{u^2}\ du = \frac{-5}{16}\frac{1}{u} + C = \frac{-5}{16}\frac{1}{\sin\theta} + C$$

From the triangle we see that $\sin\theta = \dfrac{5x}{\sqrt{25x^2+16}}$.

So $\dfrac{1}{\sin\theta} = \dfrac{\sqrt{25x^2+16}}{5x}$.



Answer: $\dfrac{-5}{16}\cdot\dfrac{\sqrt{25x^2+16}}{5x} + C = \dfrac{-\sqrt{25x^2+16}}{16x} + C$

3. This integral does not have a square root but we can still use the identity, $\sec^2\theta = 1 + \tan^2\theta$. Let $x = \tan\theta$ and then $dx = \sec^2\theta\ d\theta$

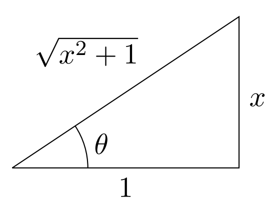
$$\int \frac{1}{(1+x^2)^2}\ dx = \int \frac{1}{(1+\tan^2\theta)^2}\sec^2\theta\ d\theta = \int \frac{\sec^2\theta}{(\sec^2\theta)^2}\ d\theta = \int \frac{1}{\sec^2\theta}\ d\theta =$$

$$= \int \cos^2\theta\ d\theta = \int \frac{1}{2}(1+\cos 2\theta)\ d\theta = \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C = \frac{1}{2}\theta + \frac{1}{4}\cdot 2\sin\theta\cos\theta + C = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$$

From the triangle we see that $\sin\theta = \dfrac{x}{\sqrt{x^2+1}}$

and $\cos\theta = \dfrac{1}{\sqrt{x^2+1}}$



Answer: $\dfrac{1}{2}\arctan(x) + \dfrac{1}{2}\dfrac{x}{\sqrt{x^2+1}}\dfrac{1}{\sqrt{x^2+1}} + C = \dfrac{1}{2}\arctan(x) + \dfrac{1}{2}\dfrac{1}{x^2+1} + C$

4. The expression under the square root matches the form of the identity, $\cos^2\theta = 1 - \sin^2\theta$, except the constant is a 4. So multiply the identity by 4 to fix the constant.

$4\cos^2\theta = 4 - 4\sin^2\theta$

we need $9x^2 = 4\sin^2\theta$ so let $3x = 2\sin\theta$ or $x = \dfrac{2}{3}\sin\theta$ and then $dx = \dfrac{2}{3}\cos\theta\ d\theta$

$$\int x^3\sqrt{4-9x^2}\ dx = \int \left(\frac{2}{3}\sin\theta\right)^3 \sqrt{4-4\sin^2\theta}\cdot\frac{2}{3}\cos\theta\ d\theta = \int \left(\frac{2}{3}\right)^4 \sin^3\theta\cos\theta\sqrt{4\cos^2\theta}\ d\theta$$

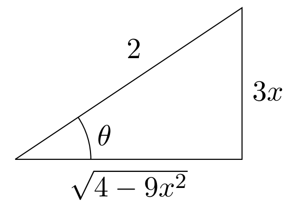
$$= \int \frac{32}{81}\sin^3\theta\cos^2\theta\ d\theta = \frac{32}{81}\int \sin^2\theta\cos^2\theta\sin\theta\ d\theta = \frac{32}{81}\int (1-\cos^2\theta)\cos^2\theta\sin\theta\ d\theta$$

Now we want to use the u-sub of $u = \cos\theta$ with $du = -\sin\theta\ d\theta$. This will give the integral

$$= \frac{-32}{81}\int (1-u^2)u^2\ du = \frac{-32}{81}\int (u^2 - u^4)\ du = \frac{-32}{81}\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + C$$

$$= \frac{-32}{81}\left(\frac{1}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta\right) + C$$

From the triangle we see that $\cos\theta = \dfrac{\sqrt{4-9x^2}}{2}$



Answer: $\dfrac{-32}{81}\left[\dfrac{1}{3}\left(\dfrac{\sqrt{4-9x^2}}{2}\right)^3 - \dfrac{1}{5}\left(\dfrac{\sqrt{4-9x^2}}{2}\right)^5\right] + C$

5. The expression under the square root matches the form of the identity, $\tan^2\theta = \sec^2\theta - 1$, except the constant is a 9. So multiply the identity by 9 to fix the constant.

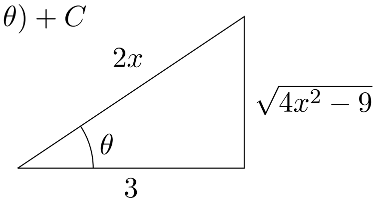
   $9\tan^2\theta = 9\sec^2\theta - 9$

   we need $4x^2 = 9\sec^2\theta$ so let $2x = 3\sec\theta$ or $x = \frac{3}{2}\sec\theta$ and then $dx = \frac{3}{2}\sec\theta\tan\theta\; d\theta$

   $$\int \frac{\sqrt{4x^2-9}}{x}\,dx = \int \frac{\sqrt{9\sec^2\theta - 9}}{\frac{3}{2}\sec\theta}\;\frac{3}{2}\sec\theta\tan\theta\;d\theta = \int \sqrt{9\tan^2\theta}\quad \tan\theta\;d\theta$$

   $$= \int 3\tan^2\theta\;d\theta = 3\int \sec^2\theta - 1\;d\theta = 3\left(\tan\theta - \theta\right) + C$$

   From the triangle we see that $\tan\theta = \dfrac{\sqrt{4x^2-9}}{3}$



   Answer: $3\left(\dfrac{\sqrt{4x^2-9}}{3} - \arccos\left(\dfrac{3}{2x}\right)\right) + C = \sqrt{4x^2-9} - 3\arccos\left(\dfrac{3}{2x}\right) + C$

6. Complete the square so we can rewrite the integral in a form that will work with trig sub.

   $$\begin{aligned} 6x - x^2 &= -x^2 + 6x \\ &= -1(x^2 - 6x) \\ &= -1\left(x^2 - 6x + 3^2 - 3^2\right) \\ &= -1\left(x^2 - 6x + 9 - 9\right) \\ &= -1\left((x-3)^2 - 9\right) \\ &= -(x-3)^2 + 9 \\ &= 9 - (x-3)^2 \end{aligned}$$

   $$\int \frac{4}{(6x-x^2)^{3/2}}\,dx = \int \frac{4}{(9-(x-3)^2)^{3/2}}\,dx$$

   The expression under the square root matches the form of the identity, $\cos^2\theta = 1 - \sin^2\theta$, except the constant is a 9. So multiply the identity by 9 to fix the constant.

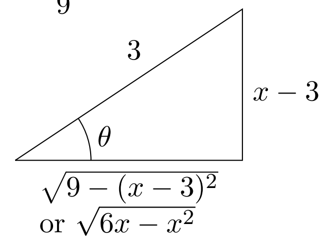
   $9\cos^2\theta = 9 - 9\sin^2\theta$

   we need $(x-3)^2 = 9\sin^2\theta$ so let $x - 3 = 3\sin\theta$ or $x = 3 + 3\sin\theta$ and then $dx = 3\cos\theta\;d\theta$

   $$\int \frac{4}{(9-(x-3)^2)^{3/2}}\,dx = \int \frac{4}{(9-9\sin^2\theta)^{3/2}}\;3\cos\theta\;d\theta = \int \frac{12\cos\theta}{(9\cos^2\theta)^{3/2}}\;d\theta$$

$$= \int \frac{12\cos\theta}{27\cos^3\theta}\, d\theta = \int \frac{4}{9\cos^2\theta}\, d\theta = \int \frac{4}{9}\sec^2\theta\, d\theta = \frac{4}{9}\tan\theta + C$$

From the triangle we see that $\tan\theta = \dfrac{x-3}{\sqrt{6x-x^2}}$



Answer: $\dfrac{4}{9}\,\dfrac{x-3}{\sqrt{6x-x^2}} + C$

7. Complete the square so we can rewrite the integral in a form that will work with trig sub.

$$\begin{aligned}
9x^2+18x+13 &= 9(x^2+2x)+13 \\
&= 9\left(x^2+2x+1^2-1^2\right)+13 \\
&= 9\left(x^2+2x+1-1\right)+13 \\
&= 9\left((x+1)^2-1\right)+13 \\
&= 9(x+1)^2-9+13 \\
&= 9(x+1)^2+4
\end{aligned}$$

$$\int \frac{1}{\sqrt{9x^2+18x+13}}\, dx = \int \frac{1}{\sqrt{9(x+1)^2+4}}\, dx$$

The expression under the square root matches the form of the identity, $\sec^2\theta = 1+\tan^2\theta$, except the constant is a 4. So multiply the identity by 4 to fix the constant.

$4\sec^2\theta = 4\tan^2\theta + 4$

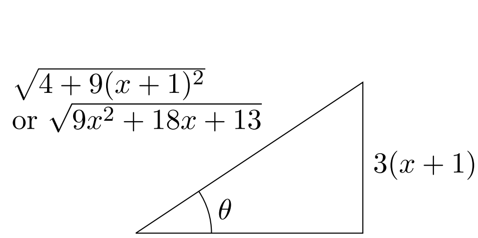
we need $9(x+1)^2 = 4\tan^2\theta$ so let $3(x+1) = 2\tan\theta$ or $x = \dfrac{1}{3}(2\tan\theta - 3)$ and then $dx = \dfrac{2}{3}\sec^2\theta\, d\theta$

$$\int \frac{1}{\sqrt{9(x+1)^2+4}}\, dx = \int \frac{1}{\sqrt{4\tan^2\theta+4}}\cdot\frac{2}{3}\sec^2\theta\, d\theta = \frac{2}{3}\int \frac{\sec^2\theta}{\sqrt{4\sec^2\theta}}\, d\theta$$

$$= \frac{2}{3}\int \frac{\sec^2\theta}{2\sec\theta}\, d\theta = \frac{1}{3}\int \sec\theta\, d\theta = \frac{1}{3}\ln|\sec\theta+\tan\theta| + C$$

From the triangle we see that $\sec\theta = \dfrac{\sqrt{4+9(x+1)^2}}{2}$

and $\tan\theta = \dfrac{3(x+1)}{2}$



Answer: $\dfrac{1}{3}\ln\left|\dfrac{\sqrt{4+9(x+1)^2}}{2} + \dfrac{3(x+1)}{2}\right| + C$

8. Complete the square so we can rewrite the integral in a form that will work with trig sub.

$$\begin{aligned}
4x - x^2 - 3 &= -x^2 + 4x - 3 \\
&= -1(x^2 - 4x) - 3 \\
&= -1\left(x^2 - 4x + 2^2 - 2^2\right) - 3 \\
&= -1\left(x^2 - 4x + 4 - 4\right) - 3 \\
&= -1\left((x-2)^2 - 4\right) - 3 \\
&= -(x-2)^2 + 4 - 3 \\
&= 1 - (x-2)^2
\end{aligned}$$

$$\int \frac{x^2}{(4x - x^2 - 3)^{3/2}}\,dx = \int \frac{x^2}{(1-(x-2)^2)^{3/2}}\,dx$$

Now procede with the trig substitution. The expression $1-(x-2)^2$ matches the form of the identity $\cos^2\theta = 1 - \sin^2\theta$.

We need $(x-2)^2 = \sin^2\theta$ so let $x - 2 = \sin\theta$ or $x = 2 + \sin\theta$ and then $dx = \cos\theta\ d\theta$

$$\int \frac{x^2}{(1-(x-2)^2)^{3/2}}\,dx = \int \frac{(2+\sin\theta)^2}{(1-\sin^2\theta)^{3/2}}\cos\theta\ d\theta = \int \frac{4 + 4\sin\theta + \sin^2\theta}{(\cos^2\theta)^{3/2}}\cos\theta\ d\theta$$

$$= \int \frac{4 + 4\sin\theta + \sin^2\theta}{\cos^3\theta}\cos\theta\ d\theta = \int \frac{4 + 4\sin\theta + \sin^2\theta}{\cos^2\theta}\ d\theta$$

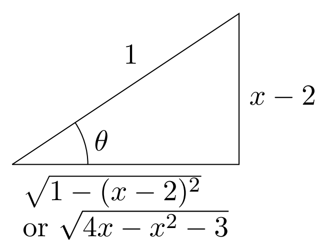
$$= \int \left(\frac{4}{\cos^2\theta} + \frac{4\sin\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}\right) d\theta = \int \left(4\sec^2\theta + 4\frac{\sin\theta}{\cos\theta}\frac{1}{\cos\theta} + \tan^2\theta\right) d\theta$$

$$= \int \left(4\sec^2\theta + 4\tan\theta\sec\theta + \sec^2\theta - 1\right) d\theta = \int \left(5\sec^2\theta + 4\tan\theta\sec\theta - 1\right) d\theta$$

$$= 5\tan\theta + 4\sec\theta - \theta + C$$

From the triangle we see that $\sec\theta = \dfrac{1}{\sqrt{4x - x^2 - 3}}$

and $\tan\theta = \dfrac{x+2}{\sqrt{4x - x^2 - 3}}$



Answer: $\dfrac{5(x+2)}{\sqrt{4x - x^2 - 3}} + \dfrac{4}{\sqrt{4x - x^2 - 3}} - \arcsin(x-2) + C$