Section 6.4: Additional Problems Solutions

1. Suppose that 4J of work is needed to stretch a spring from its natural length of 25cm to a length of 37cm. How far beyond its natural length will a force of 30N keep the spring stretched.

   Note: cm units need to be converted to m.

   We know that $4J$ of work is need to stretch the spring from its natural length to a length of 37 cm. This is a length of $0.37m - 0.25m = 0.12m$.

   $$W = 4J = \int_0^{0.12} kx\ dx = \left.\frac{kx^2}{2}\right|_0^{0.12} = \frac{k * 0.12^2}{2} - 0 = 0.0072k$$

   solving for k gives $k = \frac{5000}{9}$. The force function is $F = \frac{5000}{9}x$

   Now use F = 30N and solve for x. $30 = \frac{5000}{9}x$ or $x = 0.054m$

   Depending on the units that are being requested in the answer, the answer is 0.054 m or 5.4 cm.

2. Suppose a spring has a natural length of 3 ft and it takes 10 ft-lb to stretch it from 5 ft to 8ft.

   Step 1 is to find the force function.

   Note: the x-values are $x = 5 - 3 = 2$ and $x = 8 - 3 = 5$.

   $$W = 10ft - lb = \int_2^5 kx\ dx = \left.\frac{kx^2}{2}\right|_2^5 = \frac{25k}{2} - 2k = \frac{21k}{2}$$

   so $10 = \frac{21k}{2}$ or $k = \frac{20}{21}$

   (a) How much work is required to stretch the spring from 4 ft($x = 1$) to 7 ft($x = 4$)?

   $$W = \int_1^4 \frac{20x}{21}\ dx = \left.\frac{10x^2}{21}\right|_1^4 = \frac{150}{21}\text{ft-lbs}$$

   (b) How far beyond its natural length would a force of 3 lb keep the spring stretched?

   $F = \frac{20x}{21}$ so $3 = \frac{20x}{21}$ gives $x = 3.15$ ft. (this is beyond it natural length) or a length of 6.15 ft.

3. A bucket that weighs 8 lb and a rope that weighs 3 lb/ft are used to draw water from a well that is 70 ft deep. The bucket is filled with 60 lb of water and is pulled up at a rate of 2 ft/s, but water leaks out of a hole in the bucket at a rate of 0.4 lb/s. Find the work done W in pulling the bucket to the top of the well.

   This problem is worked as an object being moved with a non-constant force.

   First we need to know the total force when the bucket is at the bottom of the well and filled with water.

   force = bucket + rope + water = 8lb + 3lb/ft * 70ft + 60lb
   force = 8 + 210 + 60 = 278lbs

   Now we need to know how the force is decreasing for each foot the bucket is raised.

   The rope is decreasing by 3 lb/ft. The water is decreasing by $\dfrac{0.4lb/s}{2ft/s} = 0.2lb/ft$.

   Thus the force function if we considered this a point mass moving from the bottom of the well to the top of the well is

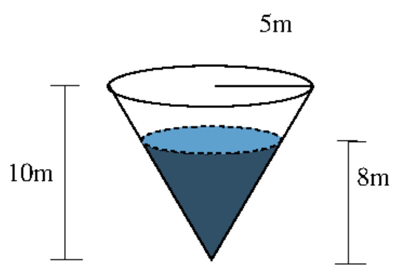
   $f = 278 - 3x - 0.2x = 278 - 3.2x$

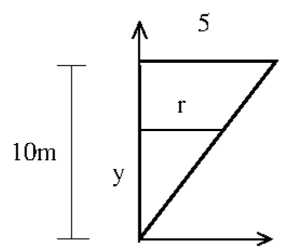
   Now integrate to find the work.

   $$W = \int_0^{70} (278 - 3.2x)\ dx = \left(278x - 1.6x^2\right)\Big|_0^{70} = 11620\text{ft-lbs}$$

4. A tank in the shape of a right circular cone of radius 5m and height 10m contains water in it to a depth of 8ft. How much work is done in pumping the water out over the top?

   Place the origin at the bottom of the cone and positive y-values go up.





   Information about the slice:

   The distance the slice moves is $d = 10 - y$

   The force for the slice is $F = \rho g V = \rho g \pi r^2 \Delta y$

   Where the relationship between r and y can be computed using similar triangles and the figure on the right. $\dfrac{r}{y} = \dfrac{5}{10}$ or $r = \dfrac{y}{2}$.

   Thus $F =]\rho g \pi \dfrac{y^2}{4} \Delta y$.

   Now transition into the integral

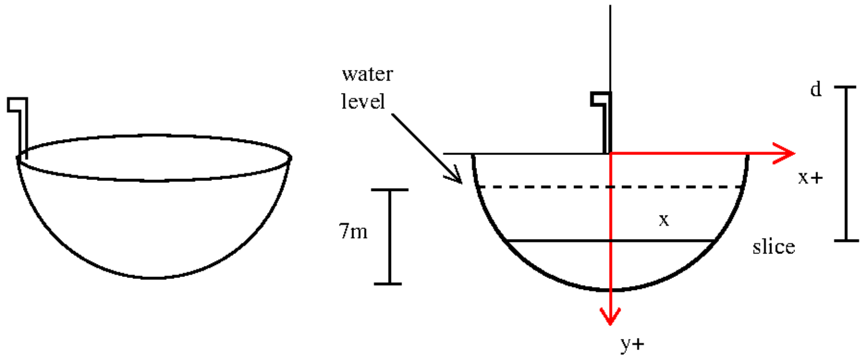
   $$W = \int_0^8 \frac{9800\pi}{4} y^2 (10 - y)\ dy = \int_0^8 2450\pi(10y^2 - y^3) dy = 2450\pi \left(\frac{10y^3}{3} - \frac{y^4}{4}\right)\Big|_0^8$$

   $W = 5.254 \times 10^6$ J

5. A Hemispherical tank has the shape shown below. The tank has a radius of 10 meters with a 2 meter spout at the rope of the tank. The tank is filled with water to a depth of 7 meters. The weight density of water is $\rho g = 9800$ N/m^3.

Set up an integral that will compute the work required to pump all of the water out of the spout. Indicate on the picture where you are placing the axis and which direction is positive.

Method 1: axis is at the top of the tank. y positive pointing down.



The distance the slice moves is $d = y + 2$.

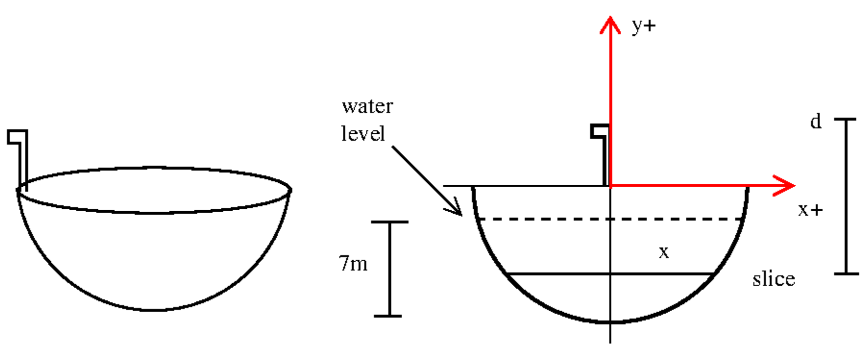
The Volume of the slice is $V = \pi x^2 \Delta y = \pi(100 - y^2)\Delta y$ since the cross section is a circle centered at the origin and thus the relationship between x and y is $x^2 + y^2 = 10^2$ or $x^2 = 100 - y^2$.

The force formula is $F = \rho g V = \rho g \pi(100 - y^2)\Delta y$.

Now transition to the integral

$$W = \int_{3}^{10} \rho g \pi (100 - y^2)(y + 2)dy$$

Method 2: axis is at the top of the tank. y positive pointing up.



The distance the slice moves is $d = 2 - y$.

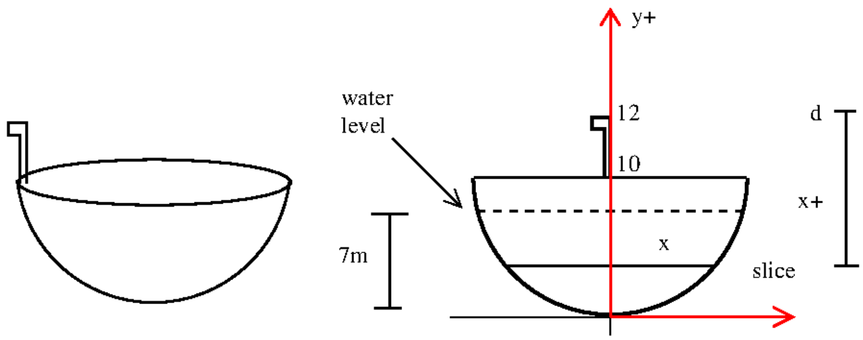
The Volume of the slice is $V = \pi x^2 \Delta y = \pi(100 - y^2)\Delta y$ since the cross section is a circle centered at the origin and thus the relationship between x and y is $x^2 + y^2 = 10^2$ or $x^2 = 100 - y^2$.

The force formula is $F = \rho g V = \rho g \pi(100 - y^2)\Delta y$.

Now transition to the integral

$$W = \int_{-10}^{-3} \rho g \pi (100 - y^2)(2 - y)dy$$

Method 3: axis is at the bottom of the tank. y positive pointing up.



The distance the slice moves is $d = 12 - y$.

The Volume of the slice is $V = \pi x^2 \Delta y = \pi(100-(y-10)^2)\Delta y$ since the cross section is a circle centered at $x = 0$ and $y = 10$ and thus the relationship between x and y is $x^2 + (y-10)^2 = 10^2$ or $x^2 = 100 - (y-10)^2$.

The force formula is $F = \rho g V = \rho g \pi(100 - (y-10)^2)\Delta y$.

Now transition to the integral

$$W = \int_0^7 \rho g \pi(100 - (y-10)^2)(12-y)dy$$