Section 2.6: Additional Problems Solutions

1. This limit has an exponential part , $a = 2\left(\frac{\pi}{2}\right)^x$. This is exponential growth since $\frac{\pi}{2} > 1$. We know that as $x \to \infty$ then $a \to \infty$ and as $x \to -\infty$ then $a \to 0$.

   (a) $\displaystyle\lim_{x\to\infty} 6 + 2\left(\frac{\pi}{2}\right)^x = \infty$

   (b) $\displaystyle\lim_{x\to-\infty} 6 + 2\left(\frac{\pi}{2}\right)^x = 6$

   (c) $\displaystyle\lim_{x\to\infty} 6 - 2\left(\frac{\pi}{2}\right)^x = -\infty$ Notice the negative sign in front of the exponential part means the final answer changes sign.

   (d) $\displaystyle\lim_{x\to-\infty} 6 - 2\left(\frac{\pi}{2}\right)^x = 6$

2. This limit has an exponential part , $a = \left(\frac{\sqrt{7}}{3}\right)^x$. This is exponential decay since $\frac{\sqrt{7}}{3} < 1$. We know that as $x \to \infty$ then $a \to 0$ and as $x \to -\infty$ then $a \to \infty$.

   (a) $\displaystyle\lim_{x\to\infty} 5 + \left(\frac{\sqrt{7}}{3}\right)^x = 5$

   (b) $\displaystyle\lim_{x\to-\infty} 5 + \left(\frac{\sqrt{7}}{3}\right)^x = \infty$

   (c) $\displaystyle\lim_{x\to\infty} \frac{2}{5 + \left(\frac{\sqrt{7}}{3}\right)^x} = \frac{2}{5}$

3. This is end behavior of a polynomial, so you need to know the degree of the polynomial.

   (a) Degree 6(even) polynomial since the highest power is $-x^6$. Since the sign on this term is negative, we know both ends point down.
   $\displaystyle\lim_{x\to\infty} 5x - 4x^6 = -\infty$

   (b) Degree 8(even) polynomial since the highest power is $4x^8$. Since the sign on this term is positive, we know both ends point up.
   $\displaystyle\lim_{x\to-\infty} 3x^7 + 4x^8 + 2 = +\infty$

   (c) Degree 3(odd) polynomial since the highest power is x^3. Since the sign on this term is positive, we know the left side points down and the right side points up.
   $\displaystyle\lim_{x\to-\infty} x^3 + 5 = -\infty$

   (d) Degree 5(odd) polynomial since the highest power is $-2x^5$. Since the sign on this term is negative, we know the left side points up and the right side points down.
   $\displaystyle\lim_{x\to-\infty} x^4 - 2x^5 + 5 = +\infty$

4. There are multiple correct answers. Here is the easiest in factored form.
   $y = \dfrac{(x-5)(7x)}{x-5)(x+3)}$

5. (a) multiply top and bottom by $\frac{1}{x^3}$. This is the highest power of x in the denominator.

$$\lim_{x\to\infty} \frac{6-3x^4}{2x^3+7} = \lim_{x\to\infty} \frac{(6-3x^4)\frac{1}{x^3}}{(2x^3+7)\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{6}{x^3}-3x}{2+\frac{7}{x^3}}$$

as $x \to \infty$ we see that $\dfrac{6}{x^3}$ and $\dfrac{7}{x^3}$ both go to zero. this means the denominator will go to the value of 2. The numerator is a bit more interesting since the $-3x \to -\infty$ as $x \to \infty$.

Thus $\displaystyle\lim_{x\to\infty} \frac{\frac{6}{x^3}-3x}{2+\frac{7}{x^3}} = -\infty$

(b) multiply top and bottom by $\frac{1}{x^3}$. This is the highest power of x in the denominator.

$$\lim_{x\to-\infty} \frac{6+3x^5}{7-2x^3} = \lim_{x\to-\infty} \frac{(6+3x^5)\frac{1}{x^3}}{(7-2x^3)\frac{1}{x^3}} = \lim_{x\to-\infty} \frac{\frac{6}{x^3}+3x^2}{\frac{7}{x^3}-2}$$

as $x \to \infty$ we see that $\dfrac{6}{x^3}$ and $\dfrac{7}{x^3}$ both go to zero. This means the denominator will go to the value of -2. Note: the sign is the important part. The numerator is a bit more interesting since the $3x^2 \to \infty$ as $x \to \infty$.

$$\lim_{x\to-\infty} \frac{\frac{6}{x^3}+3x^2}{\frac{7}{x^3}-2} = -\infty$$

6. (a) Since $x \to \infty$, we want exponental decay. i.e. e^{kx} with $k < 0$. So look at the denominator and pick the exponential that will get rid of the exponential growth in the denominator. We will multiply top and bottom by e^{-6x}. This will ensure that the denominator will never be zero in the limit process.

$$\lim_{x\to\infty} \frac{7e^{-4x}+5e^{6x}}{3e^{6x}-4e^{-3x}} = \lim_{x\to\infty} \frac{(7e^{-4x}+5e^{6x})e^{-6x}}{(3e^{6x}-4e^{-3x})e^{-6x}} = \lim_{x\to\infty} \frac{7e^{-10x}+5}{3-4e^{-9x}} =$$

Since as $x \to \infty$ we know $e^{-10x} \to 0$ and $e^{-9x} \to 0$

Thus $\displaystyle\lim_{x\to\infty} \frac{7e^{-4x}+5e^{6x}}{3e^{6x}-4e^{-3x}} = \frac{0+5}{3-0} = \frac{5}{3}$

(b) Since $x \to -\infty$, we want exponential growth. i.e. e^{kx} with $k > 0$. So look at the denominator and pick the exponential that will get rid of the exponential decay in the denominator. We will multiply top and bottom by e^{3x}. This will ensure that the denominator will never be zero in the limit process.

$$\lim_{x\to-\infty} \frac{7e^{-4x}+5e^{6x}}{3e^{6x}-4e^{-3x}} = \lim_{x\to-\infty} \frac{(7e^{-4x}+5e^{6x})e^{3x}}{(3e^{6x}-4e^{-3x})e^{3x}} = \lim_{x\to-\infty} \frac{7e^{-x}+5e^{9x}}{3e^{9x}-4}$$

Since as $x \to -\infty$ we know $e^{9x} \to 0$ and the $e^{-x} \to \infty$. This means the numerator will go to ∞ and the denominator will go to -4.

Thus $\displaystyle\lim_{x\to-\infty} \frac{7e^{-4x}+5e^{6x}}{3e^{6x}-4e^{-3x}} = -\infty$

7. The highest power of x in the denominator is x^3, so multiply the numerator and denominator by $\frac{1}{x^3}$

$$\lim_{x\to-\infty}\frac{x^2+\sqrt{5x^6+6}}{6x^3+1}=\lim_{x\to-\infty}\frac{(x^2+\sqrt{5x^6+6})\frac{1}{x^3}}{(6x^3+1)\frac{1}{x^3}}=\lim_{x\to-\infty}\frac{\frac{1}{x}+\frac{1}{x^3}\sqrt{5x^6+6}}{6+\frac{1}{x^3}}$$

Since $x\to-\infty$ we see that $\frac{1}{x^3}=\frac{1}{x}\frac{1}{x}\frac{1}{x}=\frac{-1}{\sqrt{x^2}}\frac{-1}{\sqrt{x^2}}\frac{-1}{\sqrt{x^2}}=\frac{-1}{\sqrt{x^6}}$

$$\lim_{x\to-\infty}\frac{\frac{1}{x}+\frac{1}{x^3}\sqrt{5x^6+6}}{6+\frac{1}{x^3}}=\lim_{x\to-\infty}\frac{\frac{1}{x}+\frac{-1}{\sqrt{x^6}}\sqrt{5x^6+6}}{6+\frac{1}{x^3}}=\lim_{x\to-\infty}\frac{\frac{1}{x}-\sqrt{\frac{1}{x^6}(5x^6+6)}}{6+\frac{1}{x^3}}=$$

$$\lim_{x\to-\infty}\frac{\frac{1}{x}-\sqrt{5+\frac{6}{x^6}}}{6+\frac{1}{x^3}}=\frac{0-\sqrt{5}}{6+0}=\frac{-\sqrt{5}}{6}$$

8. multiply by the conjugate $2x+\sqrt{4x^2+3x+1}$

$$\lim_{x\to\infty}\left(2x-\sqrt{4x^2+3x+1}\right)=\lim_{x\to\infty}\left(2x-\sqrt{4x^2+3x+1}\right)\frac{2x+\sqrt{4x^2+3x+1}}{2x+\sqrt{4x^2+3x+1}}=$$

$$\lim_{x\to\infty}\frac{4x^2-(4x^2+3x+1)}{2x+\sqrt{4x^2+3x+1}}=\lim_{x\to\infty}\frac{-3x-1}{2x+\sqrt{4x^2+3x+1}}=$$

now multiply top and bottom by $\frac{1}{x}$

$$\lim_{x\to\infty}\frac{(-3x-1)\frac{1}{x}}{(2x+\sqrt{4x^2+3x+1})\frac{1}{x}}=\lim_{x\to\infty}\frac{-3-\frac{1}{x}}{2+\frac{1}{x}\sqrt{4x^2+3x+1}}=$$

since $x\to\infty$ we know that $\frac{1}{x}=\frac{1}{\sqrt{x^2}}$

$$\lim_{x\to\infty}\frac{-3-\frac{1}{x}}{2+\frac{1}{\sqrt{x^2}}\sqrt{4x^2+3x+1}}=\lim_{x\to\infty}\frac{-3-\frac{1}{x}}{2+\sqrt{4+\frac{3}{x}+\frac{1}{x^2}}}=\frac{-3+0}{2+\sqrt{4+0+0}}=\frac{-3}{4}$$

9. either multiply top and bottom by $\dfrac{1}{x^5}$ or factor out the x^5 from the top and bottom and simplify.

$$\lim_{x\to\infty}\frac{3x^4+x^5+6}{7x^5+2x^3+7}=\lim_{x\to\infty}\frac{\frac{3}{x}+1+\frac{6}{x^5}}{7+\frac{2}{x^2}+\frac{7}{x^5}}=\frac{1}{7}$$

10. first combine the logarithms into a single logarithm.

$$\lim_{x\to\infty}\left[2\ln(2x+1)-\ln(2+x^2)\right]=\lim_{x\to\infty}\left[\ln(2x+1)^2-\ln(2+x^2)\right]=\lim_{x\to\infty}\ln\left(\frac{(2x+1)^2}{(2+x^2)}\right)$$

$$\lim_{x\to\infty}\ln\left(\frac{4x^2+4x+1}{2+x^2}\right)$$

Now lets look at the fraction inside the logarithm and what it does with the limit.

$$\lim_{x\to\infty}\frac{4x^2+4x+1}{2+x^2}=\lim_{x\to\infty}\frac{x^2(4+\frac{4}{x}+\frac{1}{x^2})}{x^2(\frac{2}{x^2}+1)}=\lim_{x\to\infty}\frac{4+\frac{4}{x}+\frac{1}{x^2}}{\frac{2}{x^2}+1}=4$$

Thus the final answer to the problem is

$$\lim_{x\to\infty}\ln\left(\frac{4x^2+4x+1}{2+x^2}\right)=\ln(4)$$

11. by the setup of the problem we know this uses the squeeze theorem. So we need to consider both limits.

$$\lim_{x\to\infty}\frac{12e^x-15}{4e^x}=\lim_{x\to\infty}\left(\frac{12e^x}{4e^x}-\frac{15}{4e^x}\right)=\lim_{x\to\infty}3-\frac{15}{4}e^{-x}=3$$

$$\lim_{x\to\infty}\frac{3\sqrt{x}}{\sqrt{x-1}}=\lim_{x\to\infty}3\sqrt{\frac{x}{x-1}}$$

now consider $\lim_{x\to\infty}\frac{x}{x-1}$. (multiply to and bottom by $\frac{1}{x}$).

$$\lim_{x\to\infty}\frac{x}{x-1}=\lim_{x\to\infty}\frac{1}{1-\frac{1}{x}}=1.$$

thus $\lim_{x\to\infty}\frac{3\sqrt{x}}{\sqrt{x-1}}=\lim_{x\to\infty}3\sqrt{\frac{x}{x-1}}=3\sqrt{1}=3$

Finally we see that by the squeeze theorem $\lim_{x\to\infty}f(x)=3$.