Chapter 1: The Measurement of Interest

Section 1.8: Nominal Rate of Interest and Discounts

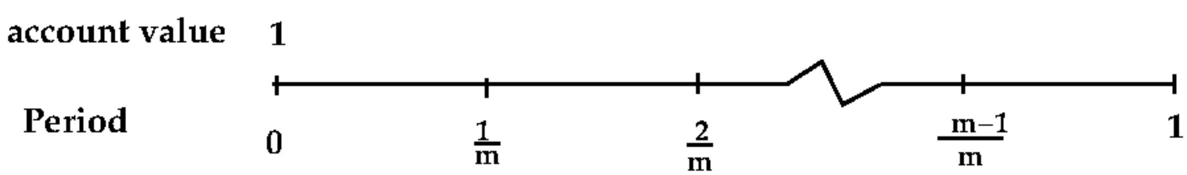
The term **effective** is used for rates of interest and discounts in which interest is paid once per measurement period.

When interest is paid more frequently that once per measurement period. These rates are called a **nominal** rates.

Example: 5% compounded quarterly
6% payable monthly
3% convertable semiannually

The **interest conversion period** is the frequency with which interest is paid and reinvested to earn additional interest.

The **nominal rate of interest** payable **m** times per period (typically a year) is denoted by $i^{(m)}$. The rate of interest for each m-th of a period (effective rate per period) is $\frac{i^{(m)}}{m}$.



NOTE: Most calculators have built-in functions that convert between a nominal rate and an effective rate.

Example: What is the annual effective rate of interest for an account with an nominal rate of 5% compounded quarterly?

Example: If $i^{(4)} = 7\%$, find the equivalent rate of $i^{(6)}$.

Example: Find the future value of \$200 invested for 5 years at 8% per annum convertible quarterly. i.e Find the future value of \$200 invested for 5 years at a nominal rate of 8% compounded quarterly.

The **nominal rate of discount** payable **m** times per period (typically a year) is denoted by $d^{(m)}$. The rate of discount for each m-th of a period is $\dfrac{d^{(m)}}{m}$.

In a similar manner as before, the present value of a payment of \$1 to be made at the end of a year is found by the formula:

$$1 - d = \left(1 - \frac{d^{(m)}}{m}\right)^m$$

Example: If $i^{(6)} = 3.4\%$, find $d^{(12)}$.

Example: Find the present value of \$7,000 to paid at the end of 5 years at 6% per annum payable in advance convertible semimonthly.

Section 1.9: Forces of Interest and Discount

Effective rates of interest: i measure interest over one measurement period.

Nominal rates of interest: $i^{(m)}$ measure interest over m-ths of a measurement period.

Force of interest: δ_t measures interest over infinitesimally small intervals of time.

Suppose $A(t)$ represents the amount function of an investment at time t where t is measured in years. Consider $A(t)$ over the interval $\left[t, t+\frac{1}{m}\right]$. Lets assume that we have a nominal rate $i^{(m)}$ for this interval.

Definition: For an investment that grows according to an accumulated amount function $A(t)$, the force of interest at time t, is defined to be

$$\delta_t =$$

δ_t is a nominal annual interest rate compounded infinitely often or compounded continuously.

δ_t is also interpreted as the instantaneous rate of growth of the investment per dollar invested at time point t.

Example: Consider the amount function $A(t) = 25\left(1+\frac{t}{4}\right)^3$. At what time is the force of interest equal to 0.5?

Example: A fund's value at time t is given by $A(t) = 7t^2 + 3t + 75$. Find δ_2

Now consider for $t_1 < t_2$: $\displaystyle\int_{t_1}^{t_2} \delta_r \, dr =$

Special Case: $t_1 = 0$ and $t_2 = t$: $$\int_0^t \delta_r \, dr = \cdots = \ln\left(\frac{A(t)}{A(0)}\right)$$

Example: An investment of \$500 accumulates at a force of interest of $\delta_t = 0.2 - 0.02t$,

(a) Find $A(3)$

(b) Find the interest earned between times 2 and 4.

Example: Using a force of interest of $\delta_t = 0.1 - 0.002t$, find the value of the account at the end of the third year, if the account has a value of \$200 at the end of the 8th year.

Derive an expression for δ_t based on

(a) simple interest at annual rate i.

(b) compound interest at annual rate i.

<u>Force of discount:</u>

We can define the force of discount, $\delta_t^{'}$, in a similar manner. Let $A(t)$ be an amount function and $d^{(m)}$ a nominal discount rate on the interval $\left[t, t+\frac{1}{m}\right]$

$A\left(t+\dfrac{1}{m}\right) = A(t) * \left(1 - \dfrac{d^{(m)}}{m}\right)^{-1}$ Solving for $d^{(m)}$ and taking a limit as $m \to \infty$ gives

$$\delta_t^{'} = \frac{-\dfrac{d}{dt}\left(A(t)\right)^{-1}}{\left(A(t)\right)^{-1}} = \frac{-\dfrac{d}{dt}\left(a(t)\right)^{-1}}{\left(a(t)\right)^{-1}}$$

Note: For simple interest:

Force of interest: $\delta_t = \dfrac{i}{1+it}$

Force of discount: $\delta_t' = \dfrac{d}{1-dt}$

Relationship of terms

$$\left(1+\frac{i^{(m)}}{m}\right)^m = (1+i) = v^{-1} = (1-d)^{-1} = \left(1-\frac{d^{(p)}}{p}\right)^{-p} = e^{\delta}$$

Section 1.10: Varying Interest

Example: Find the accumulated value of \$1,000 at the end of 15 years if the account has an effective rate of interest of 4% for the first three years, a nominal rate of 8% compounded monthly for the next three years, a nominal rate of 6% compounded quarterly for the next 4 years , an effective rate of discount of 10% for the next two years, and a force of interest of 3% for the last 3 years.

Example: Find the equivalent level effective rate of interest over the 15 year period for the above example.

Inflation and Real Rate of Return

Inflation is defined as a sustained increase in the general level of prices for goods and services. Inflation represents a loss of purchasing power.

Suppose you have \$100 and that today's price for milk is \$4 per gallon. That \$100 buys 25 gallons.

Now assume you invest \$100 for 2 years at an effective rate of 8% per year. In two years, you have $100(1+0.08)^2 = \$116.64$. You should then be able to buy $\frac{116.64}{4} = 29.16$ or 29 whole gallons of milk, assuming no inflation (i.e., assuming that the price of milk is still \$4 per gallon).

Now suppose that there was a 5% constant rate of inflation over these two years. This implies that milk now costs $4(1+0.05)^2 = \$4.41$ per gallon. Thus with your \$116.64 you can only buy $\frac{116.64}{4.41} = 26.44898$ or 26 whole gallons of milk.
In essence, our purchasing power grew from 25 gallons to 26.44898 gallons over this two year period, where both interest and inflation are taken into consideration. The **real rate of return**, i', is measured by solving the following equation:

$$25(1+i')^2 = 26.44898$$

In this example, we find that i', the real rate of return, is 2.857%. (Note that the real rate of return is NOT simply the difference between the rate of interest earned and the rate of inflation.)

If we work our way backwards from this equation, we can see how the real rate of return i' is found in general:

$$\begin{array}{rcl}
\begin{array}{l}\text{Present Number Buyable}\\ \text{Accumulated with the}\\ \text{Real Rate of Return}\end{array} & = & \text{Future number buyable}\\[2ex]
25(1+i')^2 & = & 26.44898\\[2ex]
\frac{\text{PV of Money}}{\text{Current price/item}} * (1+i')^2 & = & \frac{\text{FV of Money}}{\text{Future price/item}}\\[2ex]
\left(\frac{\$100}{\$4/\text{gallon}}\right)(1+i')^2 & = & \frac{\$116.64}{\$4.41/\text{gallon}}\\[2ex]
\left(\frac{100}{4}\right)(1+i')^2 & = & \frac{100(1+i)^2}{4(1+r)^2}\\[2ex]
(1+i')^2 & = & \frac{(1+i)^2}{(1+r)^2}
\end{array}$$

The final equation above simplifies to

$$1+i' = \frac{1+i}{1+r}$$

where i' is the real rate of return, i is the annual effective rate of interest, and r is the rate of inflation. Thus,

$$i' = \frac{1+i}{1+r} - 1$$

or equivalently,

$$i_{real} = i_R = i' = \frac{i-r}{1+r}$$

Example: The annual effective interest rate is 8% and the rate of inflation is 2%. Determine the real interest rate.

Example: Bob's account has an annual effective rate of 10% on which 30% tax is paid. if the inflation rate is 6% per year, what is the annual after-tax real rate of return?