Section 16.8: Stokes' Theorem

Stokes' Theorem: Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\Re^3$ that contains S. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F}\cdot d\mathbf{S}$$

Curve C has **positive orientation** means if you walk the curve such that the surface is to your left, then your head is in the direction of the normal vector.

Positive orientation is determined by the right-hand rule. The normal vector points in the same direction as your right thumb if the fingers on your right hand are pointing in the direction the curve is traversed.

Example: Evaluate $\int\limits_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \left\langle -y^2, x, z^2\right\rangle$ and C is the curve of intersection of the plane $y+z=3$ and the cylinder $x^2+y^2=4$. Orient C to be counter clockwise when viewed from above.

Method 1: parametrize the curve by $\mathbf{r}(\theta) = \langle 2\cos\theta, 2\sin\theta, 3-2\sin\theta\rangle$.

$\mathbf{F} = \left\langle -4\sin^2\theta, 2\cos\theta, (3-\sin\theta)^2\right\rangle$ and $\mathbf{r}'(\theta) = \langle -2\sin\theta, 2\cos\theta, -2\cos\theta\rangle$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{\theta=0}^{2\pi} \mathbf{F}\cdot\mathbf{r}'d\theta = \int_{\theta=0}^{2\pi} \left(8\sin^3\theta + 4\cos^2\theta - 2\cos\theta(3-\sin\theta)^2\right) d\theta$$

Example: Find the work performed by the force field $\mathbf{F} = \left\langle 3x^8, 4xy^3, y^2x \right\rangle$ on a particle that traverses the curve C in the plane $z = y$ consisting of 4 line segments starting at the point $(0, 0, 0)$ to $(1, 0, 0)$ to $(1, 3, 3)$ to $(0, 3, 3)$ to $(0, 0, 0)$.

Example: Use Stokes' Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ if S is the part of the hemisphere $x = \sqrt{9 - y^2 - z^2}$ that lies inside the cylinder $y^2 + z^2 = 4$, oriented in the direction of the positive $x-$axis.

$\mathbf{F} = \left\langle xy,\ xy^2z,\ z + x \right\rangle$