Section 16.7: Surface Integrals

Definition: If S is parametrized by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, then the **surface integral of f over the surface S** is

$$\iint_S f(x,y,z)dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r_u} \times \mathbf{r_v}|dA$$

where D is a region in the $uv-$plane.

Application: If the function is the density at the points of the surface then the surface integral over S computes the mass of the surface.

mass: $m = \iint_S \rho(x,y,z)dS$

Center of mass:

$$\overline{x} = \frac{1}{m}\iint_S x\rho(x,y,z)dS, \quad \overline{y} = \frac{1}{m}\iint_S y\rho(x,y,z)dS, \quad \overline{z} = \frac{1}{m}\iint_S z\rho(x,y,z)dS$$

Example: Evaluate $\iint_S xz dS$ where S is the part of the plane $3x + 2y + z = 6$ in the first octant.

Example: Compute $\iint_S y^2z^2 dS$ where S is the part of the cone $z = \sqrt{x^2+y^2}$ between the planes $z = 1$ and $z = 2$.

$x = r\cos\theta$
$y = r\sin\theta$
$z = \sqrt{r^2} = r$

$r(r,\theta) = \langle r\cos\theta,\ r\sin\theta,\ r\rangle$

$$r_r \times r_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -r\cos\theta, -r\sin\theta, r\rangle$$

$$|r_r \times r_\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{r^2+r^2} = \sqrt{2r^2} = r\sqrt{2}$$

$$\iint_S y^2z^2 dS = \int_{\theta=0}^{2\pi}\int_{r=1}^{2} r^2\sin^2(\theta)\ r^2\ r\sqrt{2}\ \ dr d\theta = \ldots \frac{21\pi\sqrt{2}}{2}$$

Example: Compute $\iint_S y^2z^2 dS$ where S is the part of the cone $z = \sqrt{x^2+y^2}$ between the planes $z = 1$ and $z = 2$.

Using $x = x \qquad y = y \qquad z = \sqrt{x^2+y^2}$

$r(x,y) = \left\langle x,\ y,\ \sqrt{x^2+y^2}\right\rangle$

$$r_x \times r_y = \left\langle \frac{-x}{\sqrt{x^2+y^2}},\ \frac{-y}{\sqrt{x^2+y^2}},\ 1\right\rangle$$

$$|r_x \times r_y| = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} = \sqrt{2}$$

$$\iint_S y^2z^2 dS = \iint_D y^2(\sqrt{x^2+y^2})^2\sqrt{2}\ \ dA =$$

$$\int_{\theta=0}^{2\pi}\int_{r=1}^{2} r^2\sin^2(\theta)\ r^2\ \sqrt{2}\ \ r\ dr d\theta = \ldots \frac{21\pi\sqrt{2}}{2}$$

Example: Compute $\displaystyle\iint_S xy\,dS$ where S is the boundary of the region enclosed by the cylinder $x^2+z^2=1$ and the planes $y=1$ and $x+y=3$

Surface integrals over vector fields.

Let S be a surface parametrized by $\mathbf{r}(u,v)$. If S has a tangent plane at every point on S (except at any boundary points), then there are two unit normal vectors at every point.

$\mathbf{n}_1 = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$ and $\mathbf{n}_2 = \dfrac{\mathbf{r}_v \times \mathbf{r}_u}{|\mathbf{r}_v \times \mathbf{r}_u|}$

The normal vector provides an orientation for S and S is called an **oriented surface**

For a surface defined by $z = g(x,y)$, then $\mathbf{n} = \dfrac{\langle -g_x, -g_y, 1\rangle}{\sqrt{1+(g_x)^2+(g_y)^2}}$

Since the $\mathbf{k}$ component is positive, this gives the upward orientation of the surface.

Note: For a closed surface, a surface that is the boundary of a solid region(volume), **positive orientation** is where the normal vectors point outward from the region.

Definition: If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the **surface integral of F over S** is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS$$

This integral is also called the **flux** of $\mathbf{F}$ across S.

Note: If S is parametrized by $\mathbf{r}(u,v)$, then $\mathbf{n} = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$

This gives $d\mathbf{S} = \mathbf{n}\, dS = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, dS = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, |\mathbf{r}_u \times \mathbf{r}_v| dA = (\mathbf{r}_u \times \mathbf{r}_v) dA$

Thus

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v) dA$$

Note: choose the cross product that gives the correct orientation for the problem.

Example: Let S be the part of the paraboloid $z = 1 - x^2 - y^2$ above the $xy-$plane with upward orientation. Find the flux of $\mathbf{F} = \langle x, y, 3z \rangle$ across S.

Example: Let S be the sphere $x^2 + y^2 + z^2 = 16$ with a positive orientation and $\mathbf{F} = \langle 0, 0, z \rangle$. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$

Example: Let S be the closed surface of a tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, i.e .the surface of the solid in the first octant that is formed by the plane $x+y+z=1$ and the three coordinate planes. Let $\mathbf{F} = \langle y, z-y, x \rangle$ and use positive orientation.

Evaluate $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$