Section 16.6: Parametric Surfaces and Their Areas

A space curve is parametrized by the vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

A surface, $z = f(x, y)$, is parametrized by a vector function of two variables.
$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ with (u, v) in region D.

Useful Parametrizations

- Surface given by $z = f(x, y)$.

Example: $z = x^2 + 4y^2$

Example: $x = z^2 + y^2$

- Surface in cylindrical coordinates

Example: $x^2 + y^2 = 9$ for $0 \leq z \leq 2$

- Surface in spherical coordinates

Example: $x^2 + y^2 + z^2 = 4$

Example: Identify the surface with the given vector equation.

$\mathbf{r}(u,v) = \left\langle u+2, 9+u^2+v^2+4u, v \right\rangle$

Example: Find the tangent plane to the surface with parametric equations given below at the point $(1,4,5)$.

$\mathbf{r}(u,v) = \left\langle u^3, v^2, u+2v \right\rangle$

Note: In the special case the surface is defined by $z = f(x,y)$ and is parametrized by $\mathbf{r}(x,y) = \langle x, y, f(x,y) \rangle$. Then a normal vector is

Example: Find a normal vector for the surface defined as $x = f(y,z)$

Definition: If a smooth parametric surface S is given by the equation $\mathbf{r}(u, v)$ and S is covered just once as (u, v) ranges throughout the parametric domain D, then the **surface area** of S is

$$A(S) = \iint_D dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

Example: Find the surface area for the surface given by $x = uv$, $y = u + v$, and $z = u - v$ where $u^2 + v^2 \leq 1$

Example: Find the surface area for the part of the plane $2x+2y+z=8$ inside the cylinder $x^2+y^2=9$.

Example: Find the surface area of the sphere $x^2+y^2+z^2=16$ between the planes $z=2$ and $z=2\sqrt{3}$.

$x = 4\sin\phi\cos\theta$
$y = 4\sin\phi\sin\theta$
$z = 4\cos\phi$

where $0 \leq \theta \leq 2\pi$ and $\frac{\pi}{6} \leq \phi \leq \frac{\pi}{3}$

$$r_\phi \times r_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4\cos\phi\cos\theta & 4\cos\phi\sin\theta & -4\sin\phi \\ -4\sin\phi\sin\theta & 4\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$r_\phi \times r_\theta = \left\langle 16\sin^2\phi\cos\theta,\ 16\sin^2\phi\sin\theta,\ 16\sin\phi\cos\phi\cos^2\theta + 16\sin\phi\cos\phi\sin^2\theta \right\rangle$$

$$r_\phi \times r_\theta = \left\langle 16\sin^2\phi\cos\theta,\ 16\sin^2\phi\sin\theta,\ 16\sin\phi\cos\phi \right\rangle$$

$$|r_\phi \times r_\theta| = \sqrt{16^2\sin^4\phi\cos^2\theta + 16^2\sin^4\phi\sin^2\theta + 16^2\sin^2\phi\cos^2\phi}$$

$$|r_\phi \times r_\theta| = \sqrt{16^2\sin^4\phi + 16^2\sin^2\phi\cos^2\phi}$$

$$|r_\phi \times r_\theta| = \sqrt{16^2\sin^2\phi(\sin^2\phi + \cos^2\phi)} = \sqrt{16^2\sin^2\phi} = 16\sin\phi$$

Note: $\sin\phi > 0$ on the given interval of ϕ.

$$S = \int_{\theta=0}^{2\pi}\int_{\phi=\pi/6}^{\pi/3} |r_\phi \times r_\theta|\, d\phi\ d\theta = \int_{\theta=0}^{2\pi}\int_{\phi=\pi/6}^{\pi/3} 16\sin\phi\ \ d\phi\ d\theta = \cdots = 16\pi(\sqrt{3}-1)$$

Example: Find the surface area of the sphere $x^2+y^2+z^2=16$ between the planes $z=2$ and $z=2\sqrt{3}$.

$r(x,y)=\left\langle x, y, \sqrt{16-x^2-y^2}\right\rangle$

if $z=2$ then this gives $x^2+y^2+4=16$ or $x^2+y^2=12$. A circle of radius $2\sqrt{3}$.

if $z=2\sqrt{3}$ then this gives $x^2+y^2+12=16$ or $x^2+y^2=4$. A circle of radius 2.

$$r_x \times r_y = \left\langle \frac{x}{\sqrt{16-x^2-y^2}}, \frac{y}{\sqrt{16-x^2-y^2}}, 1 \right\rangle$$

$$|r_x \times r_y| = \sqrt{\frac{x^2}{16-x^2-y^2}+\frac{y^2}{16-x^2-y^2}+1} = \sqrt{\frac{x^2}{16-x^2-y^2}+\frac{y^2}{16-x^2-y^2}+\frac{16-x^2-y^2}{16-x^2-y^2}}$$

$$|r_x \times r_y| = \sqrt{\frac{16}{16-x^2-y^2}} = \frac{4}{\sqrt{16-x^2-y^2}}$$

$$S = \iint\limits_D |r_x \times r_y|\, dA = \iint\limits_D \frac{4}{\sqrt{16-x^2-y^2}} dA = \int_{\theta=0}^{2\pi}\int_{r=2}^{2\sqrt{3}} \frac{4r}{\sqrt{16-r^2}}\, dr d\theta = \cdots = 16\pi(\sqrt{3}-1)$$