## Section 16.5: Curl and Divergence

**Definition:** The **del operator**, denoted $\nabla$, is defined as $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

**Definition:** If $\mathbf{F} = \langle P, Q, R \rangle$ is a a vector field on $\Re^3$ and the partial derivatives of $P$, $Q$, and $R$ all exist, then the **curl** of $\mathbf{F}$ is the vector field on $\Re^3$ defined by

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

$$\text{curl } \mathbf{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$$

Note: The curl $\mathbf{F}$ relates to rotation of a fluid at point $P$ around the axis that points in the same direction of curl $\mathbf{F}$.

Note: If the curl $\mathbf{F} = \mathbf{0}$ at point P, then $\mathbf{F}$ is called **irrotational** at P.

Example: Find the curl of the vector field $\mathbf{F} = \left\langle x^2y, yz^2, zx^2 \right\rangle$

**Theorem:** If $f$ is a function of three variables that has continuous second-order partial derivatives, then curl$(\nabla f) = \mathbf{0}$.

$\nabla f = \langle P, Q, R\rangle = \langle f_x, f_y, f_z\rangle$

curl$(\nabla f) = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\rangle$

curl$(\nabla f) = \langle f_{zy} - f_{yz},\ f_{xz} - f_{zx},\ f_{yx} - f_{xy}\rangle = \langle 0,\ 0,\ 0\rangle$

**Theorem:** If **F** is a vector field defined on all of $\Re^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = 0$, then **F** is a conservative vector field.

Example: Determine if the vector field is conservative. $\mathbf{F} = \langle zx, xy, yz\rangle$.

**Definition:** If $\mathbf{F} = \langle P, Q, R\rangle$ is a a vector field on $\Re^3$ and $P_x$, $Q_y$ and $R_z$ exist, then the **divergence of F** is the scalar function given by

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \mathbf{F}$$

Note: If **F** is a velocity field for a fluid, then div **F** at a point measures the tendency of the fluid to diverge from that point. div **F** positive(negative) means move away(towards).

Example: Compute the div **F** where $\mathbf{F} = \left\langle y^2z^3,\ 2xyz^3,\ 3xy^2z^2\right\rangle$.

**Theorem:** If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field on $\Re^3$ and $P$, $Q$, and $R$ have continuous second-order partial derivatives, then

div curl $\mathbf{F} = \text{div}\, \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle = R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0$

div curl $\mathbf{F} = 0$

Example: Is there a vector field $\mathbf{G}$ on $\Re^3$ such that curl $\mathbf{G} = \langle yz, xyz, zy \rangle$?

Example: Compute curl $\mathbf{F}$ if $\mathbf{F}$ is a vector field on $\Re^2$.

Suppose that $F = \langle P, Q \rangle$. Lets expand this to three dimensions by letting the z component, $R$, be zero.

i.e. $F = \langle P, Q, 0 \rangle$.

Thus curl$\mathbf{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle = \langle 0,\ 0,\ Q_x - P_y \rangle$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy = \iint_D Q_x - P_y dA = \iint_D \text{curl } \mathbf{F} \cdot \mathbf{k}\ dA$$