Section 16.4: Green's Theorem

The **positive orientation** of a simple closed curve C refers to a single counterclockwise traversal of C.

Green's Theorem: Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA$$

Alternate notations: When $F = \langle P, Q \rangle$ and curve given by $r(t)$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_C Pdx + Qdy = \oint_C Pdx + Qdy = \int_{\partial D} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA$$

Example: Evaluate $\oint_C x^2 y dx + x dy$ where C is the triangular path from $(0,0)$ to $(1,0)$ to $(1,4)$ to (0.0).

Example: Suppose a particle travels one revolution counter-clockwise around the unit circle under the force field $\mathbf{F}(x,y) = \langle e^x - y^3, \cos(y) + x^3 \rangle$. Find the work done by the field.

Example: Evaluate $\oint_C \left(x^2y + \frac{1}{2}y^2 + e^{\sin(x)} \right) dx + \left(xy + \frac{1}{3}x^3 + x - \arctan(y) \right) dy$ where C is the triangular path from $(0,0)$ to $(1,0)$ to $(1,4)$ to (0.0).

Area Using Line Integrals:

Since $\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA$, we need $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$

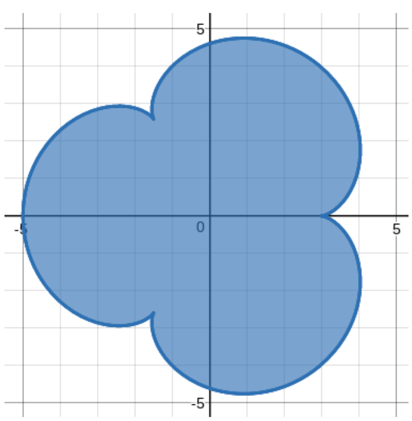
Method 1:
$P = 0$ and $Q = x$

Method 2:
$P = -y$ and $Q = 0$

Method 3:
$P = \frac{-1}{2} y$ and $Q = \frac{1}{2} x$

Example: Find the area enclosed by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Example: Setup the integral that would give the area of the shaded region shown below. The figure is created with the parametric equations: $x = 4\cos(t) - \cos(4t), \quad y = 4\sin(t) - \sin(4t)$



Example: Evaluate $\int_C x^2 y dx + x dy$ where C is the path from $(1,4)$ to $(0,0)$to $(1,0)$