Section 16.3: The Fundamental Theorem for Line Integrals

Recall the Fundamental Theorem of Calculus: $\displaystyle\int_a^b F'(x)dx = F(b) - F(a)$

Theorem: Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} =$$

Note: The line integral of a conservative vector field (∇f with potential function f) can be evaluated by knowing the endpoints of the curve.

Note: This can also be used on curves that are that are piecewise smooth.

Example: Let $f(x, y) = 3x + x^2y - y^3$. Evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \nabla f$ and C is the curve given by $\mathbf{r}(t) = \left\langle e^t \sin(t), e^t \cos(t)\right\rangle$, $0 \le t \le \pi$.

Definition: If $\mathbf{F}$ is a continuous vector field with domain D, we say that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if and only if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 with the same starting and ending points.

Note: **Line integrals of conservative vectors fields are independent of path.**

Theorem: $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D. (A closed path is one that starts and stops at the same point.)

An interpretation is that the work done by a conservative vector field as an object moves around a closed path is 0.

Question: How do we determine if a vector field is conservative and if so, can we find the potential function?

Theorem: IF $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then we have $P_y = Q_x$.

Example: Is $\mathbf{F} = \left\langle 3x^2 - 4y, 4y^2 - 2x \right\rangle$ a conservative vector field?

Definition: A **simple curve** is a curve that does not intersect itself anywhere between its endpoints. A **simply-connected region** in the plane is one that is connected and does not have holes. i.e. every simple closed curve encloses only points in the region.

Theorem: Let $\mathbf{F} = \langle P, Q \rangle$ be a vector field on an **open simply-connected region** D. Suppose that P and Q have continuous first-order derivatives and $P_y = Q_x$ throughout D. Then $\mathbf{F}$ is conservative.

Note: The above criteria to determine if a vector field is conservative works only for $\Re^2$. The criteria for a vector field in $\Re^3$ is found in section 16.5.

Example: Determine whether $\mathbf{F} = \left\langle x + y^2, 2xy + y^2 \right\rangle$ is conservative or not. If so, find a potential function.

Example: Given that $\mathbf{F} = \left\langle 4xe^z, \cos(y), 2x^2e^z \right\rangle$ is conservative. Compute $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{r}(t) = \langle \sin(t), t, \cos(t) \rangle$, for $0 \leq t \leq \dfrac{\pi}{2}$.

Example: Given $\mathbf{F} = \left\langle 2xy^3 + z^2, 3x^2y^2 + 2yz, y^2 + 2xz \right\rangle$ is conservative. Find a potential function F.