Section 16.2: Line Integrals

Reminder: In section 13.3 we discussed arc length of a space curve, $\mathbf{r}(t)$, on the interval $a \leq t \leq b$. The length of the curve, L is given by

$$L = \int_a^b ds = \int_a^b \left|\mathbf{r}'(t)\right| \; dt.$$

Line integrals on a plane:

Let C be a smooth curve defined by the parametric equations $x = x(t)$, $y = y(t)$ or by the vector function $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, for $a \leq t \leq b$.

Definition: If f is defined on a smooth curve C, as defined above, then the **line integral of f along C** is

$$\int_C f(x,y)\; ds = \lim_{||P||\to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i \qquad \text{if the limit exists.}$$

If f is a continuous function, then we can compute this line integral by

$$\int_C f(x,y) ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; dt$$

Note: This is sometimes refers to as the **line integral with respect to arc length**.

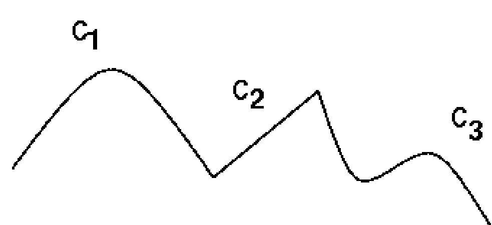
Note: The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

When $f(x,y) \geq 0$, the line integral of f along C represents the area of one side of the "fence" or "curtain" whose base is C and whose height at any point on the curve is $f(x,y)$. If $f(x,y) = 1$, then the line integral of f along C is the arc length of the curve C.

Example: Evaluate $\int_C (x^2 + y)\ ds$ where C consists of the line segment from the point $(2, 8)$ to $(0, 16)$

Example: Evaluate $\int_C (2 + x^2 y)ds$, where C is the upper half of the circle $x^2 + y^2 = 4$.

Definition: If C is a **piecewise-smooth curve**, that is C is made up a collection of smooth curves where one curve ends then the next curve begins, then line integral of f along C is defined to be the sum of the integrals of f along each smooth piece of C.



$$\int_C f(x,y)ds = \int_{C_1} f(x,y)ds + \int_{C_2} f(x,y)ds + \int_{C_3} f(x,y)ds$$

Example: Evaluate $\int_C 2y\,ds$, where C consists of C_1 of $y = x^3$ from $(0, 0)$ to $(2, 8)$ followed by the line segment from $(2, 8)$ to $(4, 2)$.

Definition: Let C be a smooth curve defined by the parametric equations $x = x(t)$, $y = y(t)$ for $a \leq t \leq b$.

The **line integral of f along C with respect to x** is $\displaystyle\int_C f(x,y)dx = \int_a^b f(x(t), y(t))\ x'(t)\ dt$

The **line integral of f along C with respect to y** is $\displaystyle\int_C f(x,y)dy = \int_a^b f(x(t), y(t))\ y'(t)\ dt$

Note: **For these integrals, the orientation of the curve, which direction is is traversed, is important.** If C and $-C$ represent traversing the same curve but in different directions, then

$$\int_{-C} f(x,y)dx = -\int_C f(x,y)dx$$

Example: Evaluate $\displaystyle\int_C y^2dx + xdy$, where C is the line segment from $(-5,-3)$ to $(3,1)$.

Example: Evaluate $\int_C y^2 dx + x dy$, where C is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(3, 1)$.

Example: Consider $f(x, y) = x^2$ and C be the smooth curve $r(t) = \langle t, 0 \rangle$ for $0 \leq t \leq 2$. Then $x(t) = t$ and $y(t) = 0$ and $x'(t) = 1\ dt$

$$\int_C f(x, y) dx = \int_0^2 t^2 * 1\ dt = \int_0^2 t^2 dt$$

Compare this to integrating $y = x^2$ on the interval $[0, 2]$ which is $\int_0^2 x^2 dx$

Example: The curve C is the line segment from $(3, 0)$ to $(3, 15)$.

Compute $\int_C (x^2 + 2y) dx$

Example: The curve C is the line segment from $(2, 5)$ to $(7, 5)$.

Compute $\int_C (x^2 + 2y) dy$

Line Integrals in Space:

Let C be a smooth curve defined by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \leq t \leq b$. The line integral of f along C is

$$\int_C f(x,y,z)\ ds = \int_a^b f(x(t), y(t), z(t))\ \ |\mathbf{r}'(t)|\ dt$$

Note: The line integral of f with respect to x, with respect to y, and with respect to z are defined in a manner similar to before.

Physical interpretation of a line integral: Let $\rho(x,y)$ represent the linear density at a point (x,y) of a thin wire shaped like the curve C.

Then the mass of the wire is $m = \displaystyle\int_C \rho(x,y)ds$

The center of mass $(\overline{x}, \overline{y})$ is

$$\overline{x} = \frac{1}{m}\int_C x\rho(x,y)ds \text{ and } \overline{y} = \frac{1}{m}\int_C y\rho(x,y)ds$$

Example: A thin wire with linear density $\rho(x,y) = 2 + x^2 y$ takes the shape of the semicircle $x^2+y^2 = 4$, $y \geq 0$. Find the mass of this wire.

Line Integrals of Vector Fields

Suppose **F** is a continuous vector field (i.e. force field). Find the work done moving a particle along the curve C given by $\mathbf{r}(t)$ for $a \leq t \leq b$.

Definition: Let **F** be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Then the **line integral of F along C** is

$$\int_c \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_c \mathbf{F} \cdot \mathbf{T}\ ds$$

Note: If the orientation of the curve is changed, i.e. C is replaced by $-C$, then the unit tangent vector is replaced by its negative. Thus

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

Example: Find the work done by the force field $\mathbf{F}$ in moving a particle along the curve C.

$\mathbf{F}(x, y, z) = \langle xy, yz, xz \rangle$

C: $\mathbf{r}(t) = \left\langle t, t^2, t^3 \right\rangle$, $0 \le t \le 1$.

Relationship between a line integral over a vector field and line integrals with respect to x, y, and z.

Let $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$ with

C defined by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \le t \le b$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \langle P, Q, R \rangle \cdot \langle dx, dy, dz \rangle = \int_C P dx + Q dy + R dz$$