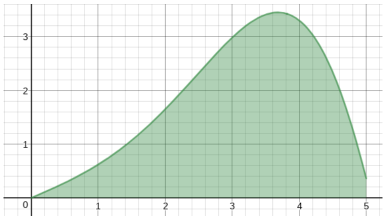
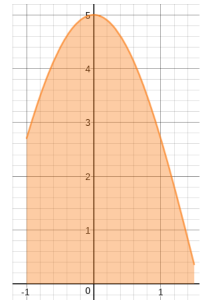
Section 15.9: Change of Variables in Multiple Integrals

From Cal 1/Cal 2 we know the following integrals are equivalent with the substitution $u = 0.1x^2 - 1$.



$$\int_0^5 x\cos(0.1x^2 - 1)dx = \int_{-1}^{1.5} 5\cos(u)du \approx 9.19483$$



Consider $\iint_R F(x,y)dA$ where R is a region in the $xy-$plane. Suppose we make the substitution $x = x(u,v)$ and $y = y(u,v)$ where x and y are functions of u and v that have continuous first-order partial derivatives. These equations give a **transformation** that will take a region S in the $uv-$plane and map it into a region R in the $xy-$plane, also called the image of S.

This will give $\displaystyle\iint_R F(x,y)dA = \iint_S F(x(u,v), y(u,v))dA$

In order to find the region S that transforms into region R, we need the transformation be one-to-one. (This means no two points (u_1, v_1) and (u_2, v_2) map to the same point (x_1, y_1). Also needed is that as the boundary of S is traversed once, then the boundary of R will also be traversed only once.

Example: A transformation is defined by the equations $x = u^2 - v^2, \quad y = 2uv$.

Find the image of the square $S = \{(u,v) | 0 \le u \le 2,\ 0 \le v \le 1\}$

Definition: The **Jacobian** of the transformation T given by $x = x(u, v)$ and $y = y(u, v)$ is

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \qquad \text{or} \qquad J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$$

Example: Find the Jacobian of the transformation defined by $x = u^2 - v^2, \quad y = 2uv.$

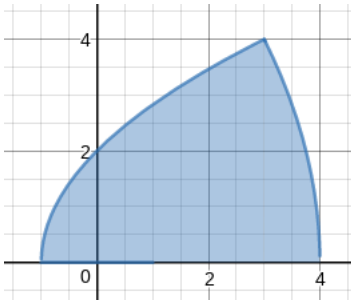
Example: Find the Jacobian of the transformation defined by $x = r\cos\theta, \quad y = r\sin\theta.$

Change of Variable in a Double Integral Suppose T is a one-to-one transformation, where the substitutions have continuous first-order partial derivatives, whose Jacobian is nonzero and that maps a region S in the $uv-$plane onto a region R in the $xy-$plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Then

$$\iint\limits_R f(x,y)dA = \iint\limits_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

Example: Let R be the region bounded by the $x-$axis and the parabolas $y^2 = 64-16x$ and $y^2 = 4+4x$, $y \geq 0$. Use the change of variables $x = u^2 - v^2$ and $y = 2uv$ to evaluate

$$\iint\limits_R y \, dA$$



Region R

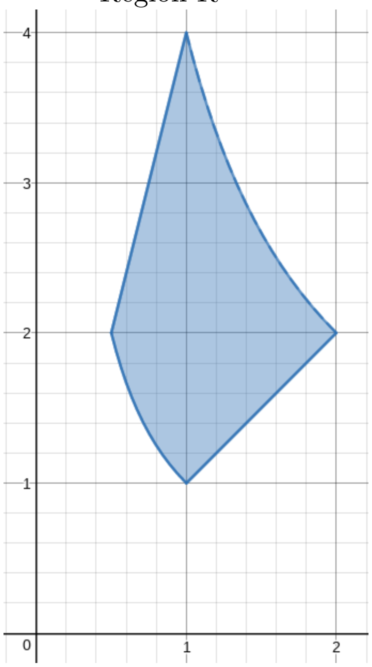
Example: Let R be the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$. Evaluate by making the given transformation.

$\displaystyle\iint_R y + 3 \, dA$ with $x = 2u$ and $y = 3v$

Example: Let R be the region in the first quadrant bounded by the lines $y = x$ and $y = 4x$ and the hyperbolas $xy = 1$ and $xy = 4$. Evaluate by making the given transformation.

$$\iint_R xy \, dA \text{ with } x = u/v \text{ and } y = v$$

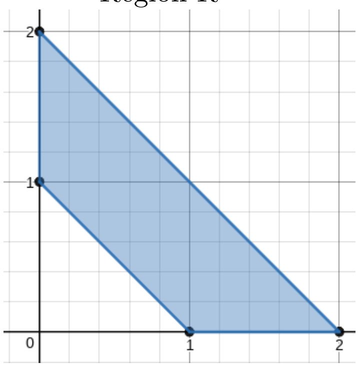
Region R



Example: Let R be the region in the $xy-$plane bounded by the vertices $(0,1)$, $(0,2)$, $(2,0)$, and $(1,0)$. Evaluate

$$\iint\limits_R e^{(y-x)/(y+x)} dA$$

Region R



Triple Integrals

Given the transformation $x = x(u, v, w)$, $y = y(u, v, w)$ and $z = z(u, v, w)$ then the Jacobian is the following 3×3 determinant.

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

With a hypotheses similar to the double integral change of variables we have the following for the triple integral.

$$\iiint\limits_R f(x, y)dV = \iiint\limits_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dudvdw$$

Example: Find the Jacobian for the transformation.

$$x = \rho \sin\phi \, \cos\theta \qquad y = \rho \sin\phi \, \sin\theta \qquad z = \rho \cos\phi$$

We compute the Jacobian as follows:

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin\phi \cos\theta & -\rho \sin\phi \sin\theta & \rho \cos\phi \cos\theta \\ \sin\phi \sin\theta & \rho \sin\phi \cos\theta & \rho \cos\phi \sin\theta \\ \cos\phi & 0 & -\rho \sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho \sin\phi \sin\theta & \rho \cos\phi \cos\theta \\ \rho \sin\phi \cos\theta & \rho \cos\phi \sin\theta \end{vmatrix} - \rho \sin\phi \begin{vmatrix} \sin\phi \cos\theta & -\rho \sin\phi \sin\theta \\ \sin\phi \sin\theta & \rho \sin\phi \cos\theta \end{vmatrix}$$

$$= \cos\phi \, (-\rho^2 \sin\phi \cos\phi \sin^2\theta - \rho^2 \sin\phi \cos\phi \cos^2\theta)$$

$$- \rho \sin\phi \, (\rho \sin^2\phi \cos^2\theta + \rho \sin^2\phi \sin^2\theta)$$

$$= -\rho^2 \sin\phi \cos^2\phi - \rho^2 \sin\phi \sin^2\phi = -\rho^2 \sin\phi$$

Since $0 \leqslant \phi \leqslant \pi$, we have $\sin\phi \geqslant 0$. Therefore

$$\left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} \right| = \left| -\rho^2 \sin\phi \right| = \rho^2 \sin\phi$$