Section 15.8: Triple Integrals in Spherical Coordinates

Spherical Coordinates:
A Cartesian point (x, y, z) is represented by (ρ, θ, ϕ) in the Spherical Coordinate System where $\rho \geq 0$ and $0 \leq \phi \leq \pi$.

Example: Find the spherical coordinates for the points $(-1, \sqrt{3}, 2)$ and $(-1, \sqrt{3}, -2)$

Example: Write the equations in spherical coordinates.

A) $x^2 + y^2 + z^2 = 25$

B) $z = 12 - 4x^2 - 4y^2$

C) $z = \sqrt{3x^2 + 3y^2}$

Triple Integrals in Spherical Coordinates

In this coordinate system, the equivalent of a box is a spherical wedge
$E = \{(\rho, \theta, \phi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$

where $a \geq 0$, $\beta - \alpha \leq 2\pi$, and $d - c \leq \pi$

$$\iiint\limits_E f(x, y, z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\ \rho^2 \sin\phi\ d\rho d\theta d\phi$$

Note: Spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region.

Example: Evaluate $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\ dz dy dx$

Example: Find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

Example: Convert the triple integral to spherical.

$$\int_{x=0}^{5} \int_{y=-\sqrt{25-x^2}}^{0} \int_{z=5-\sqrt{25-x^2-y^2}}^{5+\sqrt{25-x^2-y^2}} (x^2+y^2+z^2)^{1.5}\ dzdydx$$

Example Find the volume that is inside the sphere $x^2 + y^2 + z^2 = 4$ and above the plane $z = \sqrt{3}$.