Section 15.6: Triple Integrals

Let B be a rectangular box such that $B = \{(x, y, z) | a \le x \le b, c \le y \le d, r \le z \le s\}$, i.e. $B = [a, b] \times [c, d] \times [r, s]$.

Definition: The **triple integral** of $f(x, y, z)$ over the box B is

$$\iiint_B f(x,y,z)dV = \lim_{||P||\to 0} \sum_i \sum_j \sum_k f(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}$$

if this limit exists.

Note: The volume of solid E is given by $\iiint_E 1\ dV$.

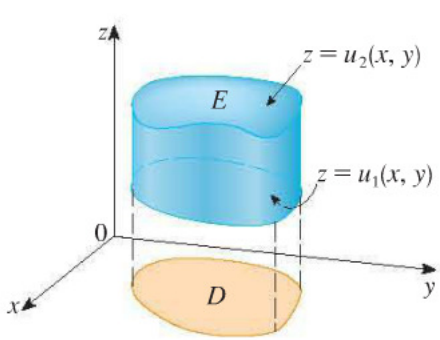
Fubini's Theorem for Triple Integrals: If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

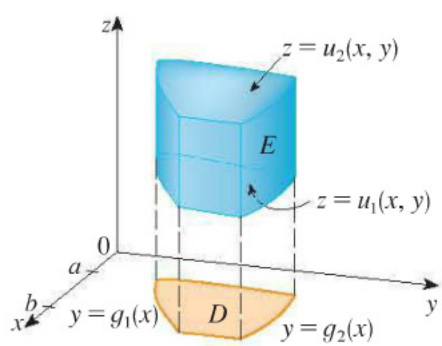
$$\iiint_B f(x,y,z)dV = \int_{z=r}^{s}\int_{y=c}^{d}\int_{x=a}^{b} f(x,y,z)\ dxdydz = \int_{x=a}^{b}\int_{z=r}^{s}\int_{y=c}^{d} f(x,y,z)\ dydzdx$$

Triple Integral over General Regions:

When using a non-rectangular solid, we consider the projection (image) the solid makes on the different coordinate planes.

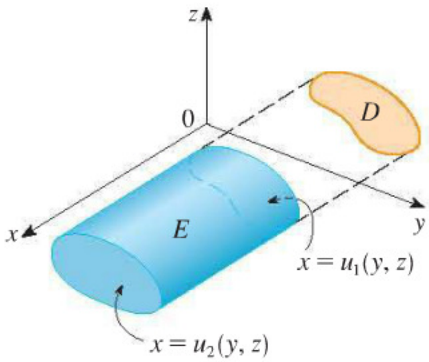
A **Type I** region is the solid $E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$





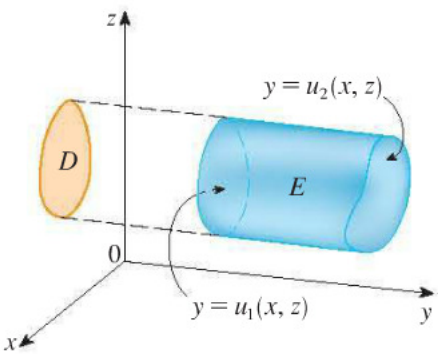
$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\right] dA$$

A **Type 2** region is the solid $E = \{(x,y,z) | (y,z) \in D, u_1(y,z) \leq x \leq u_2(y,z)\}$



$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx \right] dA$$

A **Type 3** region is the solid $E = \{(x,y,z) | (x,z) \in D, u_1(x,z) \leq y \leq u_2(x,z)\}$



$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy \right] dA$$

Example: Given E is the solid bounded by the plane $x + y = 2$ and the cylinder $y^2 + z^2 = 1$ in the first octant. Setup the triple integral with different projections on the different coordinate planes.

$$\iiint\limits_E z \, dV$$

Example: Given E is the solid bounded by the planes $z = 3 - x$, $z = 0$, $y = 0$, and $y = 2x$.

Rewrite $\displaystyle\iiint_E f(x, y, z)\ dV$ as 6 different interated integrals.

Example: Given E is the solid bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$. Compute

$$\iiint\limits_E \sqrt{x^2 + z^2}\, dV$$

Applications of Triple Integrals:
Given a solid E with density function $\rho(x, y, z)$.

mass: $m = \iiint\limits_E \rho(x, y, z)\ dV$

moments about the yz coordinate plane: $M_{yz} = \iiint\limits_E x\ \rho(x, y, z)\ dV$

moments about the xz coordinate plane: $M_{xz} = \iiint\limits_E y\ \rho(x, y, z)\ dV$

moments about the xy coordinate plane: $M_{xy} = \iiint\limits_E z\ \rho(x, y, z)\ dV$

center of Mass: $(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right)$