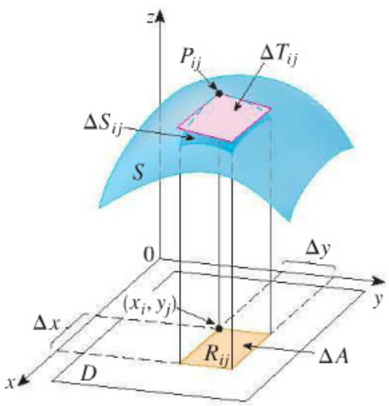
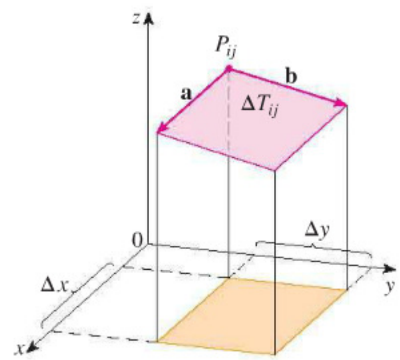
Section 15.5: Surface Area

Let S be a surface with equation $z = f(x, y)$. Assume that this surface is above the xy-plane and the domain D of f is a rectangular region. Let R_{ij} be a rectangular sub-partition of D where (x_i, y_j) is the corner of R_{ij} that is closest to the origin.

Notice from the figure, that the section of tangent plane, ΔT_{ij} at the point $P_{ij}(x_i, y_j, f(x_i, y_j))$ over the region R_{ij} will approximate the surface area on that region of the domain. Thus $A(S) \approx \sum_{i=1} \sum_{j=1} \Delta T_{ij}$





Let **a** and **b** be vectors that start at point P_{ij} and lie along the edge of ΔT_{ij}.

Thus $\mathbf{a} = \langle \Delta x, 0, f_x(x_i, y_i)\Delta x \rangle$ and $\mathbf{b} = \langle 0, \Delta y, f_y(x_i, y_i)\Delta y \rangle$ and the area of $\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}|$.

Now $\mathbf{a} \times \mathbf{b} = \langle -f_x(x_i, y_j)\Delta x\Delta y, \quad -f_y(x_i, y_j)\Delta x\Delta y, \quad \Delta x\Delta y \rangle$ Since $\Delta x\Delta y = \Delta A$ we get

$\mathbf{a} \times \mathbf{b} = \langle -f_x(x_i, y_j)\Delta A, \quad -f_y(x_i, y_j)\Delta A, \quad \Delta A \rangle$ which gives

$\Delta T_{ij} = \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1} \;\; \Delta A$

and $A(S) \approx \sum_{i=1} \sum_{j=1} \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1} \;\; \Delta A$

Definition: The area of the surface with equation $z = f(x, y)$ over the region D where f_x and f_y are continuous is given by

$$A(S) = \iint_D \sqrt{[f_x]^2 + [f_y]^2 + 1}\, dA$$

Example: Find the surface area of the part of the surface $z = 3x + y^2$ that lies above the triangle region in the $xy-$plane with vertices $(0, 0)$, $(0, 2)$, and $(2, 2)$.

Example: Find the surface area of the paraboloid given by $z = 10 - x^2 - y^2$ for $z \geq 1$.