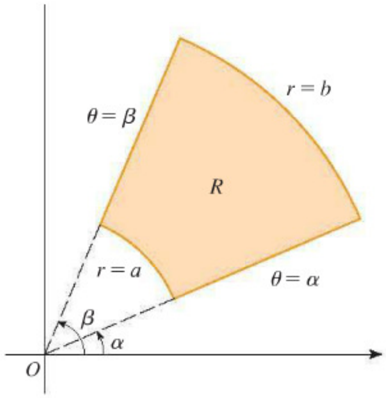
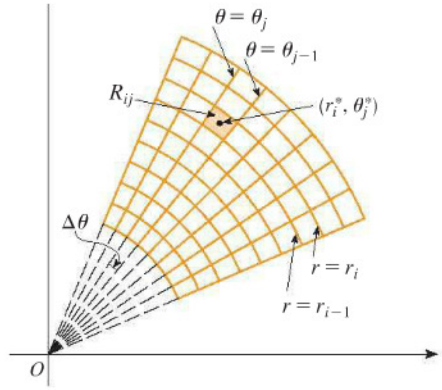
Section 15.3: Double Integrals in Polar Coordinates

Example: Evaluate $\iint\limits_D \arctan\left(\frac{y}{x}\right) dA$ where $D = \{(x,y) \mid 1 \leq x^2 + y^2 \leq 4,\ x \leq y \leq x\sqrt{3},\ x \geq 0\}$.





The center of the polar subrectangle $R_{ij} = \{(r,\theta)|r_{i-1} \le r \le r_i,\, \theta_{j-1} \le \theta \le \theta_j\}$

has polar coordinates: $r_i^* = \frac{1}{2}(r_{i-1} + r_i)$ and $\theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j)$

Note: the area a sector of a circle with radius r and angle θ is $\dfrac{1}{2}r^2\theta$.

$\Delta A_{ij} = \dfrac{1}{2}r_i^2\Delta\theta_j - \dfrac{1}{2}r_{i-1}^2\Delta\theta_j = \dfrac{1}{2}\left(r_i^2 - r_{i-1}^2\right)\Delta\theta_j$

$\Delta A_{ij} = \dfrac{1}{2}\left(r_i + r_{i-1}\right)\left(r_i - r_{i-1}\right)\Delta\theta_j = r_i^*\Delta r_i\Delta\theta_j$

thus $dA = r\ drd\theta$

Change to Polar Coordinates in a Double Integral: If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $c \le \theta \le d$, where $0 \le d - c \le 2\pi$, then

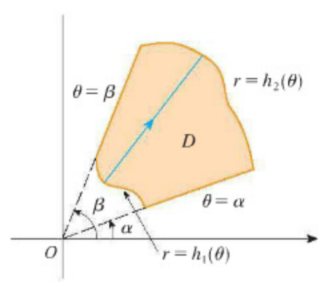
$$\iint\limits_R f(x,y)dA = \int_c^d \int_a^b f(r\cos\theta, r\sin\theta)\ \ r\ drd\theta$$

If f is continuous on a polar region of the form

$D = \{(r,\theta)|c \le \theta \le d, h_1(\theta) \le r \le h_2(\theta)\}$

then

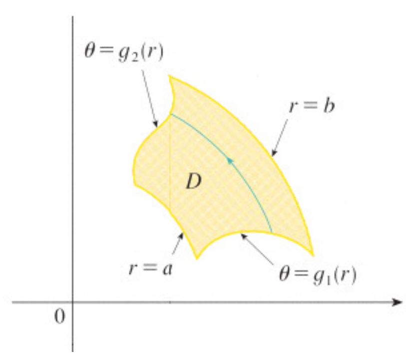
$$\iint\limits_D f(x,y)dA = \int_c^d \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\ r\ drd\theta$$



If $D = \{(r,\theta)|a \le r \le b, g_1(r) \le \theta \le g_2(r)\}$

then

$$\iint\limits_D f(x,y)dA = \int_a^b \int_{g_1(r)}^{g_2(r)} f(r\cos\theta, r\sin\theta)\ r\ d\theta dr$$



Example: Compute $\iint\limits_D y \, dA$ where D is the upper half of the cardioid: $r = 1 + \cos\theta$.

Example: Find the area of the region inside the circle $r = 4\sin\theta$ and out side the circle $r = 2$.

Example: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the $xy-$plane and inside the cylinder $x^2 + y^2 = 2x$.