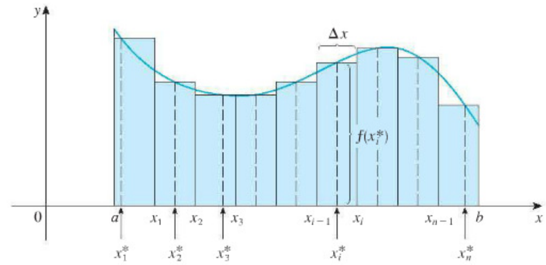
Section 15.1: Double Ingegrals over Rectangles

For a function of one variable, we define the integral in the following fashion. If $f(x)$ is defined on a closed interval $[a, b]$, we partition the interval $[a, b]$ into n equally spaced subintervals. Then

$$= \int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

If $f(x) \geq 0$ then this integral represent area.

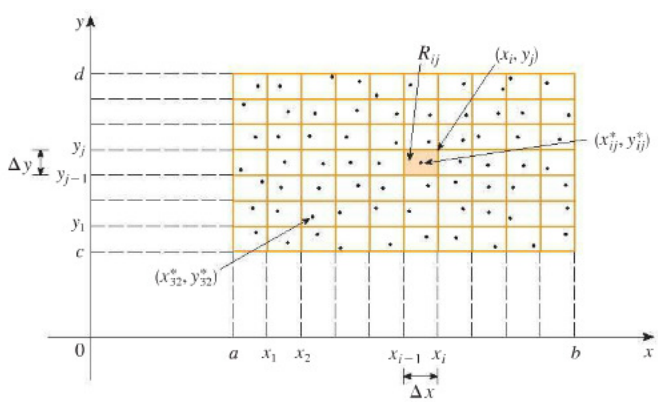


Definition of the double integral

Let $f(x, y)$ be a function of two variable defined on a closed rectangle R. Assume(for now) that $f(x, y) \geq 0$.

Define $R = [a, b] \times [c, d] = \{(x, y) \in \Re^2 |\ a \leq x \leq b, c \leq y \leq d\}$.

Lets partition both of the intervals $[a, b]$ into n equally spaced subintervals and $[c, d]$ into m equally spaced subintervals. This converts the rectangular region into a grid as shown int he picture.



Now choose a point (x_i^*, y_j^*) in the region R_{ij}. Then the **double integral** of f over the rectangle r is

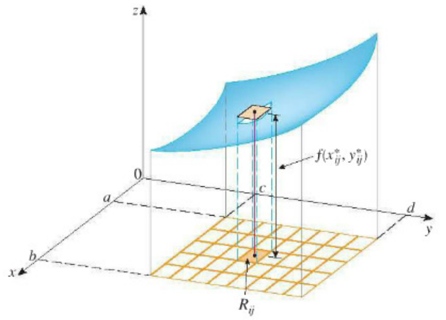
$$\iint_R f(x, y)\ dA = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*)\Delta A$$

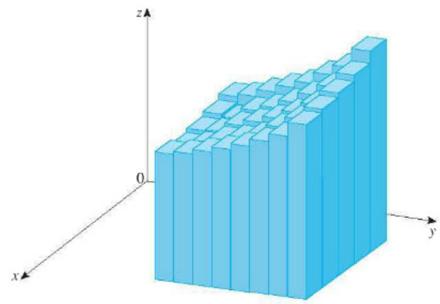
if this limit exists.

Note: Another notation is $\iint_R f(x, y)\ dA = \iint_R f(x, y) dxdy$

Theorem: If $f(x, y) \geq 0$ and f is continuous on the rectangle R, then the volume V of the solid S that lies above R and under the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\, dA$$





Example: Evaluate the integral $\iint_R \sqrt{1 - x^2}\, dA$

where $R = [-1, 1] \times [-3, 5]$ by identifying it as a volume of a solid.

Iterated Integrals

Definition: Suppose f is a function of two variables that is integrable over the rectangle $R = [a, b] \times [c, d]$.

The **partial integration** of f with respect to x is $\int_a^b f(x, y)dx$ and means that y is held fixed and f is integrated with respect to x. This result is a function of the variable y: $A(y) = \int_a^b f(x, y)dx$.

If we now integrate A with repsect to the variable y we get

$$\int_c^d A(y)dy = \int_{y=c}^{d} \left[\int_{x=a}^{b} f(x, y)dx \right] dy = \int_{y=c}^{d} \int_{x=a}^{b} f(x, y)dxdy$$

The integral above is called an **iterated integral**. Similarly, we can get the following.

$$\int_{x=a}^{b} \left[\int_{y=c}^{d} f(x, y)dy \right] dx = \int_{x=a}^{b} \int_{y=c}^{d} f(x, y)dydx$$

Example: Evaluate the following.

(A) $\displaystyle\int_{x=0}^{3}\int_{y=1}^{2} 6x^2y\ dydx$

(B) $\displaystyle\int_{1}^{2}\int_{0}^{3} 6x^2y\ dxdy$

Fubini's Theorem: If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

In the case where $f(x,y) = g(x)h(y)$ then

$$\iint_R f(x,y)dA = \int_a^b \int_c^d g(x)h(y)dydx = \int_a^b g(x)dx \int_c^d h(y)dy$$

Example: If $R = [1, 8] \times [0, \pi]$, evaluate

$$\iint_R y\cos(xy)dA$$

Example: Find the volume of the solid S that is bounded by the elliptic paraboloid $z = 6x^2 + y^2 + 1$, the planes $x = 3$ and $y = 4$ and the three coordinate planes.