Section 14.7: Maximum and Minimum values

Definition: A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b). The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geq f(a, b)$ for all (x, y) in such disk, $f(a, b)$ is a **local minimum value**.

Note: The word local is sometimes replaced with the word relative.

Theorem: If f has a **local extremum** (that is, a local maximum or minimum) at (a, b) and the first-order partial derivatives of f exists there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$

Note: If the graph of f has a tangent plane at a local extremum, then the tangent plane is horizontal.

Definition: A point (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or one of these partial derivatives does not exist, is called a **critical point** of f.

Second Derivative Test: Suppose the second partial derivatives of f are continuous in a disk with center (a, b), and suppose that (a, b) is a critical point of f. Let

$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$

(a) If $D > 0$ and $f_{xx} > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx} < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is a sadle point.
(d) If $D = 0$ then the test gives no information.

Example: Find and classify the critical values of $f(x, y) = y^3 - 6y^2 - 2x^3 - 6x^2 + 48x + 20$

Example: Find and classify the critical values of $f(x,y) = x^3 + 6xy - 2y^2$

Example: Find and classify the critical values of $f(x,y) = 1 + 2xy - x^2 - y^2$

Example: The base of a rectangular tank with volume of 540 cubic units is made of slate and the sides are made of glass. If slate costs five times as much as glass(per unit area), find the dimensions of the tank which minimize the cost of the materials.

Absolute Maximum and Absolute Minimum

Definition: A function f has a **absolute maximum** at (a,b) if $f(a,b)$ is the largest function value for the domain of f. Similarly, f has a **absolute minimum** at (a,b) if $f(a,b)$ is the smallest function value for the domain of f.

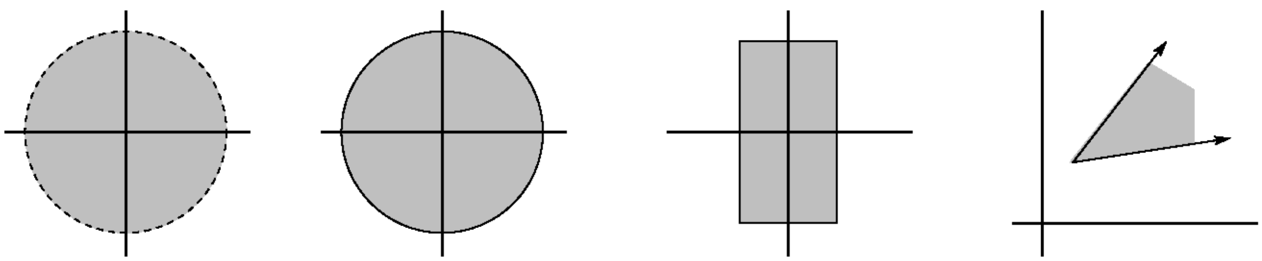
Extreme Value Theorem for Functions of One Variable: If f is continous on a closed interval $[a,b]$, then f has an absolute maximum and an absolute minimum value. These are found by evaluating critical points and the endpoints of the interval.

Extreme Value Theorem for Functions of Two Variables: If f is continuous on a closed and bounded set D in $\Re^2$, then f attains an absolute maximum value $f(x_1,y_1)$ and an absolute minimum value $f(x_2,y_2)$ at some points (x_1,y_1) and (x_2,y_2) in D.

Definition: A **closed set** in $\Re^2$ is one that contains all of its boundry points.

Definition: A **bounded set** in $\Re^2$ is one that is contained in some disk.

$\{(x,y)|x^2+y^2<4\}$ $\{(x,y)|x^2+y^2\le 4\}$ $\{(x,y)||x|\le 1, |y|\le 2\}$



To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

(1) Find the values of f at the critical points in D.
(2) Find the extreme values of f on the boundry of D.
(3) The largest of the values is the absolute maximum value; the smallest is the absolute minimum value.

Example: Find the absolute maximum/absolute minimum of f on the set D.

$D = \{(x,y) |\ 0 \leq x \leq 3, \quad -2 \leq y \leq 4-2x\}$

$f(x,y) = x^2 + xy + 2y^2 - 3x + 2y$

Example: Find the absolute max for $f(x, y) = xy$ on the set D.

$$D = \left\{ (x, y) \,\middle|\, \frac{x^2}{16} + y^2 \leq 1 \right\}$$