

Section 14.6: Directional Derivatives and the Gradient Vector

Recall that for $f(x, y)$, the first partial f_x represent the rate of change of f in the x direction and f_y represents the rate of change of f in the y direction. In other words, f_x and f_y represent the rate of change of f in the direction of the unit vectors \mathbf{i} and \mathbf{j} respectively.

Definition: The **directional derivative** of f at (x_o, y_o) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$, denoted by $D_{\mathbf{u}}f(x_o, y_o)$, is

$$D_{\mathbf{u}}f(x_o, y_o) = \lim_{h \rightarrow 0} \frac{f(x_o + ha, y_o + hb) - f(x_o, y_o)}{h}$$

if this limit exists.

Note: This shows that $f_x(x_o, y_o) = D_{\mathbf{i}}f(x_o, y_o)$ and $f_y(x_o, y_o) = D_{\mathbf{j}}f(x_o, y_o)$.

Theorem: If f is a differentiable function of x and y , then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$$

Note: If the unit vector \mathbf{u} makes an angle θ with respect to the positive x -axis, then we can write

$$\mathbf{u} = \langle \cos \theta, \sin \theta \rangle \quad \text{and} \quad D_{\mathbf{u}}f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta$$

Example: Find the rate of change of $f(x, y) = x^2 + 2xy - 3y^2$ at the point $(1, 2)$ in the direction indicated by the angle $\theta = \frac{\pi}{4}$.

Definition: If f is a function of two variables x and y , then the **gradient** of f , denoted **grad** f or ∇f , is the vector function defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

Note: ∇f which is read "del f ".

Example: Find the gradient and the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = \langle 2, 5 \rangle$.

Definition: The gradient and the directional derivative of a function $f(x, y, z)$ with unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$\nabla f(x, y, z) =$$

$$D_{\mathbf{u}}f(x, y, z) =$$

Example: Find the directional derivative of the function $f(x, y, z) = z^4 - x^3y^2$ at the point $P(1, 3, 2)$ in the direction of $Q(2, 4, 3)$.

The directional derivatives at a point P for a function f gives the rates of changes of f in all possible directions. This leads to the question: In which of these directions does f change the fastest and what is the maximum rate of change?

Theorem: Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f$ is $|\nabla f|$ and it occur when \mathbf{u} has the direction as the gradient vector.

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} =$$

Example: If $f(x, y) = xe^y$, in what direction does f have the maximum rate of change at the point $P(2, 0)$? What is the maximum rate of change of f ?

Tangent Planes to Level surfaces

Suppose S is a surface with equation $F(x, y, z) = k$. Let $P(x_o, y_o, z_o)$ be a point on the surface.

Let C be any curve on the surface going through the point P and defined by the vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ with $\mathbf{r}(t_o) = \langle x_o, y_o, z_o \rangle$.

Combining the above information gives $F(x(t), y(t), z(t)) = k$. Now if F and \mathbf{r} are differentiable, then we can use the chain rule to get the following.

$$F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt} = 0$$

$$\langle F_x, F_y, F_z \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = 0$$

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

Thus ∇F at point P is perpendicular to the tangent vector $\mathbf{r}'(t_o)$ to any curve C on the surface S passing through the point P .

This means that ∇F at point P is a normal vector for the tangent plane.

Thus the tangent plane to the surface $F(x, y, z) = k$ at point $P(x_o, y_o, z_o)$ is

$$F_x(x_o, y_o, z_o)(x - x_o) + F_y(x_o, y_o, z_o)(y - y_o) + F_z(x_o, y_o, z_o)(z - z_o) = 0$$

What is a direction vector for the normal line to the surface at the point $P(x_o, y_o, z_o)$?

Example: Find the equation of the tangent plane and the normal line to the surface at the point $(1, 2, 1)$

$$4x^2 + y^2 + 9z^2 = 17$$

In a similar manner, the gradient vector $\nabla F(x_o, y_o)$ is perpendicular to the level curves $f(x, y) = k$ at the point (x_o, y_o) .

Consider the topographical map of a hill and let $f(x, y)$ represent the elevation at the point (x, y) . Draw a curve of steepest ascent.

