Section 14.5: Chain Rule

Chain rule for functions of a single variable: If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Example: Let $z = x^y$ with $x = t^3$ and $y = \sin(t)$. Compute $z'(t) = \dfrac{dz}{dt}$.

Chain Rule (Case 1): Suppose the $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \qquad \text{or} \qquad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Example: Let $z = x^y$ with $x = t^3$ and $y = \sin(t)$. Compute $z'(t) = \dfrac{dz}{dt}$

Example: Let $z = \ln(x + y^2)$ with $x = \sqrt{1 + t^2}$ and $y = e^{3t}$. Compute $z'(t)$

Example: The radius of a right circular cone is increasing at a rate of 2.1 in/s while its height is decreasing at a rate of 1.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in and the height is 140 in?

Chain Rule (Case 2): Suppose the $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ and all first partials of x and y exists. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example: $z = \sin(x)\cos(y)$, where $x = (s-t)^2$ and $y = s^2 - t^2$. Compute z_s and z_t.

Example: If $u = x^2y + y^3z^2$ where $x = rse^t$, $y = rt^3 + s^2$ and $z = rs\sin(t)$, find u_s when $(r, s, t) = (1, 2, 0)$.

Example: Suppose f is a differentiable function of x and y, and $g(a,b) = f(e^a + \sin(b), e^a + \cos(b))$. Use the table of values to calculate $g_a(0,0)$ and $g_b(0,0)$.

	f	g	f_x	f_y
$(0,0)$	4	5	10	20
$(1,2)$	8	9	7	6

Implicit Differentiation: Suppose that an equation $F(x,y) = 0$ defines y implicitly as a differentiable function of x. i.e. $y = f(x)$ and $F(x, f(x)) = 0$ for all x in the domain of $f(x)$. Find $\dfrac{dy}{dx}$.

Example: Find $\dfrac{dy}{dx}$ if $x^3 + y^3 = 6x^2y^4$

Example: Suppose that z is given implicitly as a function $z = f(x, y)$ by an equation $F(x, y, z) = 0$. Find z_x and z_y.

Example: If $x^4 + y^3 + z^2 + xye^z = 10$. Find

(a) z_x

(b) x_y