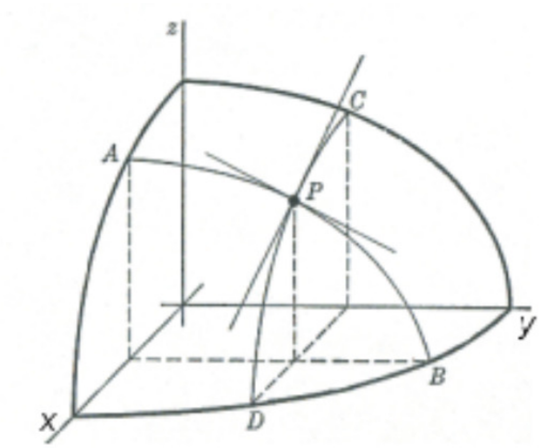
Section 14.4: Tangent Planes and Differentials

Definition: Suppose the surface S has the equation $z = f(x, y)$, where f has continuous first partials, and let $P(a, b, c)$ be a point on the surface. If C_1 and C_2 be the curves obtained by intersecting the planes $x = a$ and $y = b$ with the surface, then T_1 and T_2 are the respective tangent lines to the curves at point P. The **tangent plane** to the surface S at the point $P(a, b, c)$ is defined to be the plane that contains both of the tangent lines at point P.



Theorem: An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(a, b, c)$ or $P(a, b, f(a, b))$ is

The normal vector for this plane is

The **normal line** is the line though the point $P(a, b, c)$ that is perpendicular to the tangent plane.

Example: Find an equation of the tangent plane to the graph of the function $z = 3x^2 + y^4$ at the point $(2, 1, 13)$.

Example: Find an equation of the tangent plane $f(x, y) = \ln(5x + 2y)$ at the point $(-1, 3, 0)$. Also find a formula for the normal line at this point.

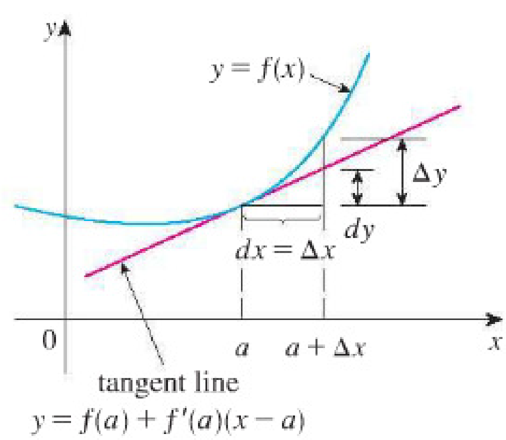
Differentials:

In Cal I we had for $y = f(x)$ the differentials dy and dx defined as

$$dy = f'(x)dx$$

We saw that for the point (a, f(a)) and $dx = \Delta x$ we found that

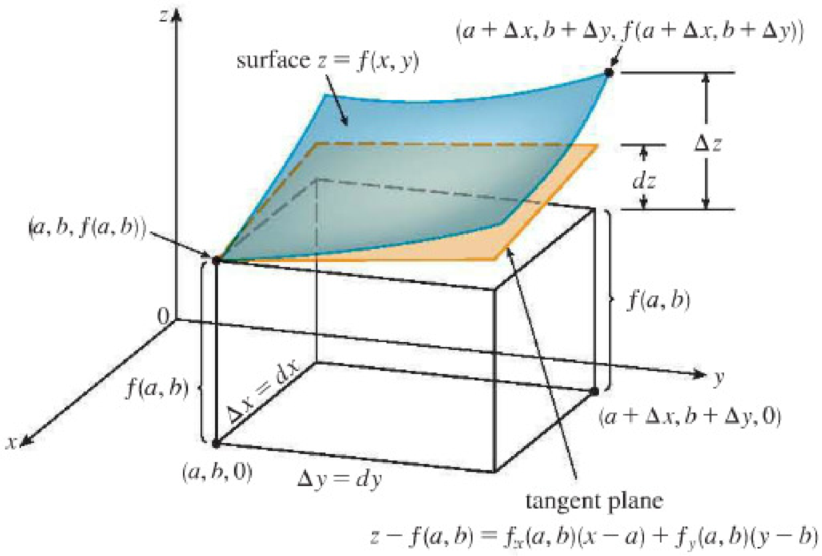
$$f(a + \Delta x) = f(a) + \Delta y \approx f(a) + dy$$



Definition: Consider a function of two variables $z = f(x, y)$. Let Δx and Δy be increments of x and y, respectively.

- Then **differentials dx and dy** are independent variables and $dx = \Delta x$ and $dy = \Delta y$
- The **differential dz**, also called the **total differential**, is a dependent variable and is defined by $dz = f_x(x, y)dx + f_y(x, y)dy$

Note: $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$ and when Δx and Δy are small and the partials are both continuous, then $\Delta z \approx dz$.



The linearization, $L(x, y)$, of the function $f(x, y)$ at the point (a, b) is
$L(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

Example: Use differentials to find an approximate value for $\sqrt{1.03^2 + 1.98^3}$

Example: Find the differential(i.e. the total differential) of the function $w = x^5y^3 + x^2z^4$.

Example: Estimate the amount of material in a closed can (right circular cylinder) with a radius of 3 in and a height of 8 in if the material of the can is 0.04 in thick.

Example: The dimensions of a closed rectangular box are measured as 90cm, 70cm, and 60cm with a possible error of 0.3cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.

Definition: A function $f(x, y)$ is **differentiable** at (a, b) if its partial derivatives f_x and f_y exist and are continuous at (a, b).

Note: Polynomial and rational functions are differentiable on their domains.

Example: Find an equation of a tangent plane for the surface at the point $(1, 1, 1)$, if it is know the two space curves $\mathbf{r}(t)$ and $\mathbf{g}(s)$ are both both on the surface and they both go though the point $(1, 1, 1)$

$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$

$\mathbf{g}(s) = \langle 1 + 2s, 1 + s - s^2, 1 - s + s^2 - s^3 \rangle$