Section 13.4: Motion in Space:

Let $\mathbf{r}(t)$ is a **position function** of a particle at time t. Then

- The **velocity function** is $\mathbf{r}'(t) = \mathbf{v}(t)$ and speed is $|\mathbf{v}(t)| = v$.
- The **acceleration function** is $\mathbf{r}''(t) = \mathbf{v}'(t) = \mathbf{a}(t)$.
- The **unit tangent vector** at t is defined to be $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$
- The **unit normal vector** at t is defined to be $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$

Now playing with functions and doing lots of algebra you get the following.

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \dfrac{\mathbf{v}(\mathbf{t})}{v(t)}$ which gives $\mathbf{T} = \dfrac{\mathbf{v}}{v}$ or $\mathbf{v} = v\mathbf{T}$

since $\kappa = \dfrac{|\mathbf{T}'|}{|\mathbf{r}'|} = \dfrac{|\mathbf{T}'|}{v}$ gives $|\mathbf{T}'| = v\kappa$

also $\mathbf{N} = \dfrac{\mathbf{T}'}{|\mathbf{T}'|}$ or $\mathbf{T}' = |\mathbf{T}'|\mathbf{N} = v\kappa\mathbf{N}$

Thus $\mathbf{a} = \mathbf{v}' = v'\mathbf{T} + v\mathbf{T}' = v'\mathbf{T} + \kappa v^2\mathbf{N}$

another way of expressing this is $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$

where $a_T = v'$ and $a_N = \kappa v^2$

after lots more algebra(and fun) we get

$$a_T = v' = \frac{\mathbf{v}\cdot\mathbf{a}}{v} = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

$$a_N = \kappa v^2 = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}\,|\mathbf{r}'(t)|^2 = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$

Example: A particle moves with position function $\mathbf{r}(t) = \langle t^2, t^2, t^3\rangle$. Find the tangential and the normal components of acceleration.

$\mathbf{r}'(t) = \langle 2t, 2t, 3t^2\rangle$

$\mathbf{r}''(t) = \langle 2, 2, 6t\rangle$

$|\mathbf{r}'(t)| = \sqrt{8t^2 + 9t^4}$

$\mathbf{r}'(t)\times\mathbf{r}''(t) = \langle 6t^2, -6t^2, 0\rangle$

$$a_T = v' = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}}$$

$$a_N = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{6t^2\sqrt{2}}{\sqrt{8t^2 + 9t^4}}$$