## Section 13.3: Arc Length and Curvature

In Cal II, the arc length of a two-dimensional smooth curve that is only traversed once on an interval $I$ was given by

$$L = \int_I ds \text{ or } L = \int_I \sqrt{[x'(t)]^2 + [y'(t)]^2}\ dt$$

This can be extended to a space curve. If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ on the interval $a \le t \le b$, then the length of the curve is given by

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\ dt \qquad \text{or} \qquad L = \int_a^b |\mathbf{r}'(t)|\ dt.$$

Note: A curve $\mathbf{r}(t)$ is called smooth on an interval if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on the interval. A smooth curve has no sharp corners or cusps, i.e. the tangent vector has continuous movement. The arc length formula holds for smooth and piecewise-smooth curves.

Example: Find the length of the arc for $\mathbf{r}(t) = \langle 3t, 2\sin(t), 2\cos(t) \rangle$ from the point $(0, 0, 2)$ to $(6\pi, 0, 2)$.

**Definition:** The **arc length function**, $s$, is $s(t) = \displaystyle\int_a^t |\mathbf{r}'(u)|\, du$.

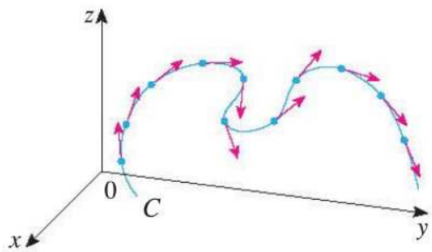
The arc length $s$ is called the **arc length parameter.**

Example: Find the arc length function for $\mathbf{r}(t) = \langle 1, t^2, t^3 \rangle$ from the point $(1, 0, 0)$ in the direction of increasing $t$.

Example: Reparametrize the curve $\mathbf{r}(t) = \langle 1 + 2t,\ 3 + t,\ -5t \rangle$ with respect to arc length measured from the point where $t = 0$ in the direction of increasing $t$.

## Curvature

**Definition:** The **curvature**, $\kappa$, of a curve is defined to be the magnitude of the rate of change of the unit tangent vector with respect to the arc length is given by $\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$




Unit tangent vectors at equally spaced points on $C$

**Theorem** The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = |\mathbf{r}''(s)|$$

Example: Find the curvature of $\mathbf{r}(t) = \left\langle -\sqrt{2}\sin t,\ \cos t,\ \cos t \right\rangle$.

Example: Find the curvature of $\mathbf{r}(t) = \left\langle 1+t, 1-t, 3t^2 \right\rangle$

Note: The **unit normal vector** is defined as $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$ and the

**binormal vector** is defined as $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$. $\mathbf{B}(t)$ is also a unit vector.

The plane determined by $\mathbf{N}(t)$ and $\mathbf{B}(t)$ is called the **normal plane**.
The plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$ is called the **osculating plane**.