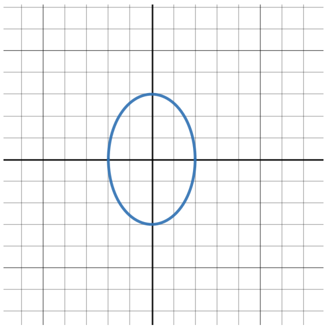
Section 12.6: Quadratic Surfaces

Review of graphs.

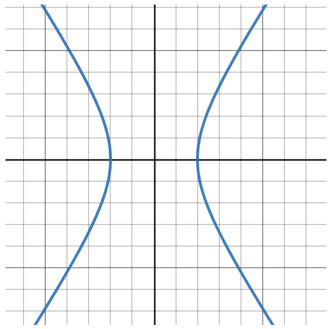
Ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

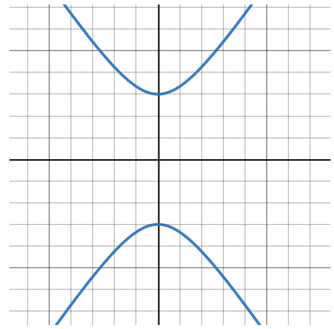


Hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



A second-degree equation in three variables x, y, and z may be expressed in one of two standard forms

$$Ax^2 + By^2 + Cz^2 + E = 0 \qquad \text{or} \qquad Ax^2 + By^2 + Cz = 0$$

where A, B, C, E are constants. To sketch the graph of a quadratic surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These curves are called **traces** or **cross-sections** of the surface.

Quadratic surfaces can be grouped into 5 categories: **quadratic cylinders**(cylindrical surfaces from 12.1 notes), **ellipsoids**, **hyperboloids**, **cones**, and **paraboloids**.

For the following examples, assume that $a > 0$, $b > 0$, and $c > 0$.

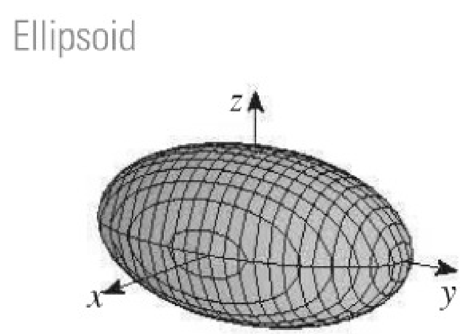
Ellipsoid:

standard equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

intercepts: $(\pm a, 0, 0)$, $(0, \pm b, 0)$, and $(0, 0, \pm c)$

cross-sections: (when they exist)
parallel to $xy-$plane($z = k$): ellipse
parallel to $xz-$plane($y = k$): ellipse
parallel to $yz-$plane($x = k$): ellipse

Note: If $a = b = c$ the figure is a sphere. If only two of the constants are equal then the figure is an ellipsoid with the trace involving the two constants being a circle.

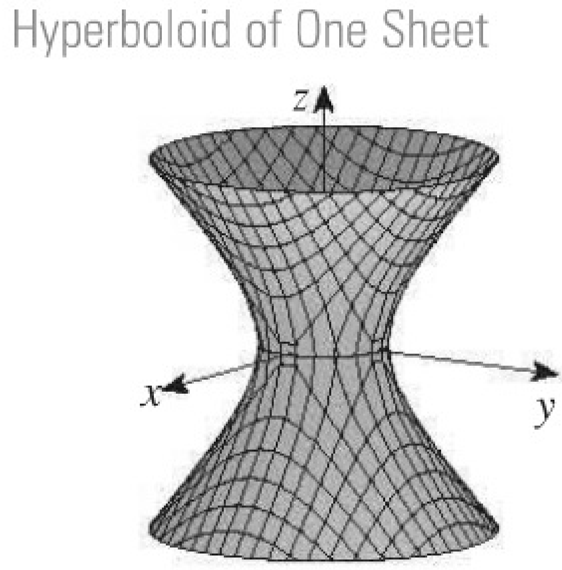


Hyperboloids:

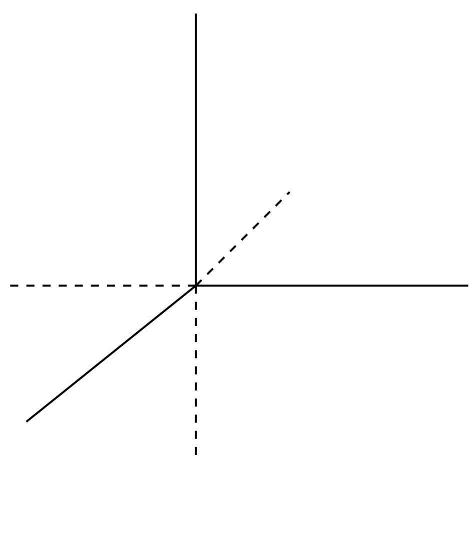
Hyperboloid of one sheet.
standard equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

cross-sections:
parallel to $xy-$plane($z = k$): ellipse
parallel to $xz-$plane($y = k$): hyperbola
parallel to $yz-$plane($x = k$): hyperbola

Note the axis of the hyperboloid corresponds to the variable whose coefficient is negative.



Example: Sketch the graph of $x^2 - \frac{y^2}{9} + z^2 = 1$

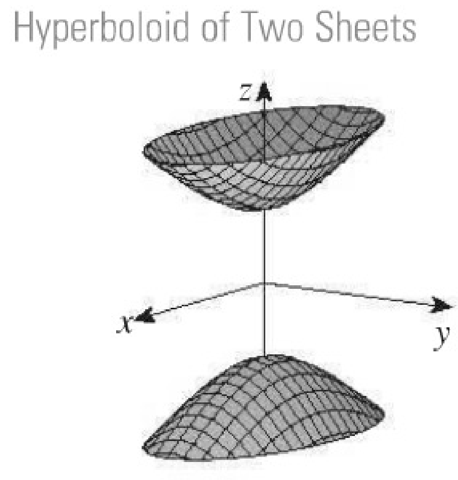


Hyperboloids:

Hyperboloid of two sheets.
standard equation: $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

cross-sections:
parallel to $xy-$plane($z = k$): ellipse (when they exist)
parallel to $xz-$plane($y = k$): hyperbola
parallel to $yz-$plane($x = k$): hyperbola

Note the axis of the hyperboloid corresponds to the variable whose coefficient is positive.

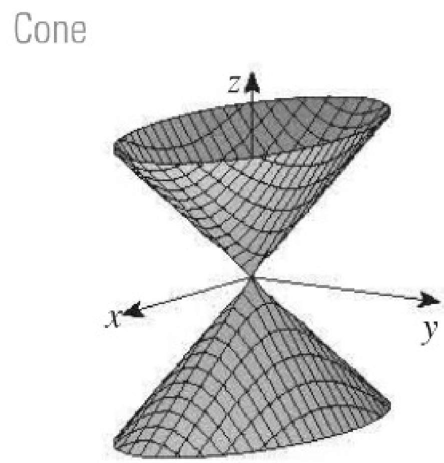


Cones:
standard equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$

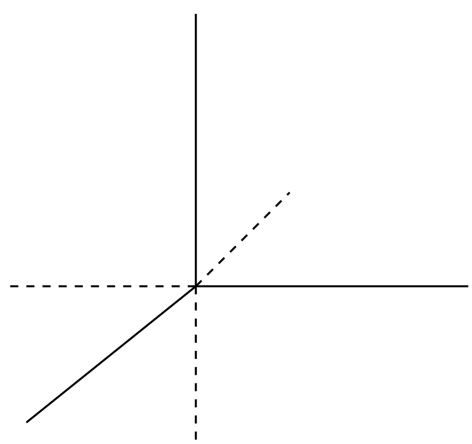
or $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

Note: If $a = b$ the we say we have a circular cone.

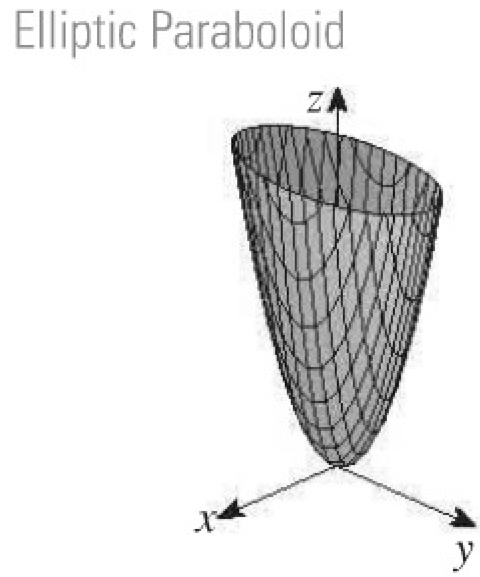
cross-sections:
parallel to $xy-$plane($z = k$): ellipse
parallel to $xz-$plane($y = k$): hyperbola for $K \neq 0$, 2 lines if $k = 0$
parallel to $yz-$plane($x = k$): hyperbola for $K \neq 0$, 2 lines if $k = 0$



Example: Sketch the graph of $z^2 = x^2 + y^2$



Paraboloids:



Elliptic paraboloid

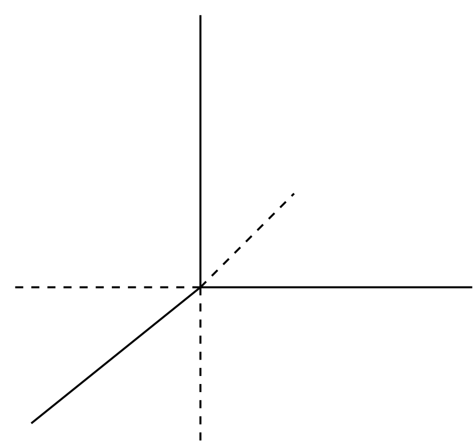
standard equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$

Note: If $a = b$ the we say we have a circular paraboloid.

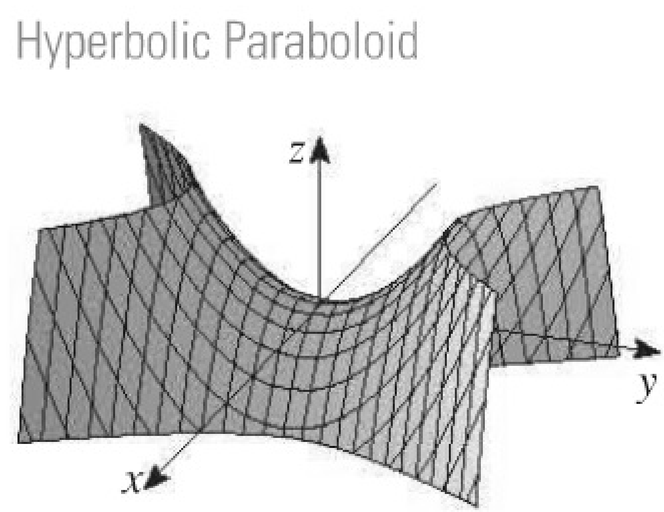
cross-sections:
parallel to $xy-$plane$(z = k)$: ellipse for $k > 0$
parallel to $xz-$plane$(y = k)$: parabola
parallel to $yz-$plane$(x = k)$: parabola

Note the axis of the paraboloid corresponds to the variable raised to the first power.

Example: Sketch the graph of $z = \frac{x^2}{4} + \frac{y^2}{9}$



Paraboloids:



hyperbolic paraboloid

standard equation: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$

cross-sections:
parallel to $xy-$plane$(z = k)$: hyperbola for $k > 0$
parallel to $xz-$plane$(y = k)$: parabola
parallel to $yz-$plane$(x = k)$: parabola

Note the axis of the paraboloid corresponds to the variable raised to the first power.