Section 12.4: The Cross Product

Reviewing the Determinate

The determinate of a 2x2 matrix is computed by

$$\begin{vmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{vmatrix} = ad - bc$$

The determinate of a 3x3 matrix is computed by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

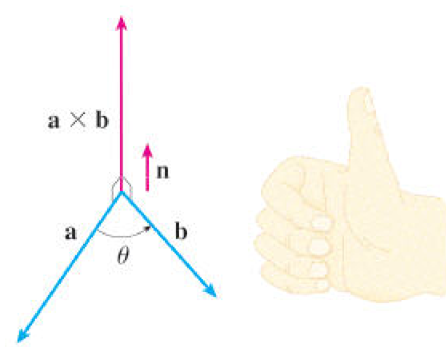
Example: Find the determinate of this matrice.

$$\begin{vmatrix} 1 & 3 & 4 \\ 5 & 0 & 2 \\ -3 & 6 & 7 \end{vmatrix}$$

Definition: If **a** and **b** are two nonzero three-dimensional vectors, the **cross product** of **a** and **b** is the vector

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}| \sin(\theta))\, \mathbf{n}$$

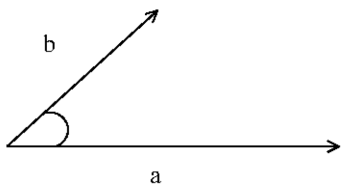
where θ is the angle, $0 \leq \theta \leq \pi$, between **a** and **b** and **n** is a **unit vector** perpendicular to both **a** and **b** and whose direction is given by the **right-hand rule**: If the fingers of your right hand curls through the angle θ from **a** to **b**, then your thumb points in the direction of **n**.



Note: $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$

Note: Two non-zero vectors,**a** and **b**, are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$

Geometric Interpretation:



Properties of the Cross Product: If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and d is a scalar, then

- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
- $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a}) = -\mathbf{b} \times \mathbf{a}$
- $(d\mathbf{a}) \times \mathbf{b} = d(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (d\mathbf{b})$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
- $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

Definition: If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, \quad a_3b_1 - a_1b_3, \quad a_1b_2 - a_2b_1 \rangle$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

Example: Compute the following for the vectors $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, -5, 6 \rangle$.

A) $\mathbf{a} \times \mathbf{b}$

B) $\mathbf{b} \times \mathbf{a}$

C) $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$

Example: Find a vector orthogonal to the plane determined by the points $A(1, 2, 3)$, $B(4, 6, 8)$, and $C(15, 2, -5)$

Example: Find the area of the parallelogram with vertices: $P(1, 1, 2)$, $Q(6, 1, 2)$, $R(4, 5, 5)$, and $S(9, 5, 5)$

Example: Find the area of the triangle determined by the points $P(1, 1, 2)$, $Q(6, 1, 2)$, and $R(4, 5, 5)$.

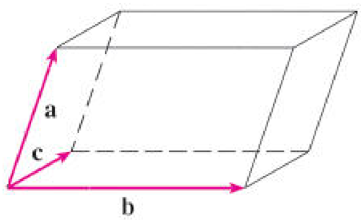
Definition: If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ are vectors, then the **scalar triple product** is given by

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \mathbf{k} \right)$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Note: The geometric interpretation of scalar triple product is that its magnitude is the volume of the parallelepiped formed by the vectors: **a**, **b**, and **c**.



Example: Compute a scalar triple product of these vectors: $\mathbf{a} = \langle 1, 2, 3 \rangle$, $\mathbf{b} = \langle 4, 5, 6 \rangle$, and $\mathbf{c} = \langle 2, 7, 5 \rangle$ Are these vectors co-planer?

Example: Determine if these points are co-planer: $A(4,-3,1)$, $B(6,-4,7)$, $C(1,2,2)$, and $D(0,1,11)$