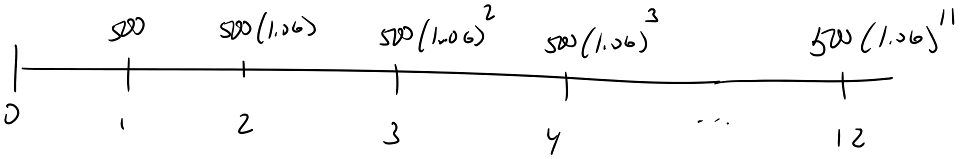
Reminder: $1 + r + r^2 + r^3 + \ldots + r^{n-1} = \dfrac{1-r^n}{1-r}$

In annuities with payments varying in geometric progression, the underlying concept is that the payments follow a compound rate of increase or decrease.

Example: Find the present value of an annuity-immediate with 12 payments in which the first payment is 500 and each subsequent payment is 6% greater than the preceding payment. The effective rate per period is 10%.

$k = 6\%$

$i = 10\%$



$1.06v = \dfrac{1.06}{1.10}$

$$PV = 500v + 500(1.06)v^2 + 500(1.06)^2 v^3 + \cdots + 500(1.06)^{11} v^{12}$$

$$= 500v\left[1 + 1.06v + (1.06)^2 v^2 + \cdots + (1.06)^{11} v^{11}\right]$$

$$= 500v\left[\frac{1-(1.06v)^{12}}{1-1.06v}\right] = \frac{500}{1.10}\left[\frac{1-\left(\frac{1.06}{1.10}\right)^{12}}{1-\frac{1.06}{1.10}}\right]$$

$$= 500\left[\frac{1-\left(\frac{1.06}{1.10}\right)^{12}}{1.10-1.06}\right] = 500\left[\frac{1-\left(\frac{1.06}{1.10}\right)^{12}}{.10-.06}\right] = PV$$

$$= 4485.65$$

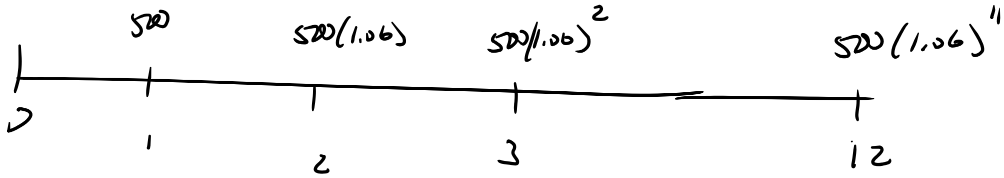
Immediate n pmts of \$R initial pmt. Increase/decrease by $k\%$ eff. interest rate of $i\%$

$$PV = R\left[\frac{1-\left(\frac{1+k}{1+i}\right)^n}{i-k}\right]$$

$$FV = PV \cdot (1+i)^n = R\left[\frac{(1+i)^n - (1+k)^n}{i-k}\right]$$

Pg 2:method #2

Example: Find the present value of an annuity-immediate with 12 payments in which the first payment is 500 and each subsequent payment is 6% greater than the preceding payment. The effective rate per period is 10%.



$$PV = 500v + 500(1.06)v^2 + 500(1.06)^2 v^3 + \cdots + 500(1.06)^{11} v^{12}$$

$$= \frac{500}{1.06}\left[1.06v + (1.06)^2 v^2 + (1.06)^3 v^3 + \cdots + (1.06)^{12} v^{12}\right]$$

Let $u = 1.06v$

$$= \frac{500}{1.06}\left[u + u^2 + u^3 + \cdots + u^{12}\right]$$

$$= \frac{500}{1.06}\, a_{\overline{12}|j}$$

j is the Rate of Interest where u is the discount for j

$$u = \frac{1}{1+j}$$

$$1.06v = \frac{1}{1+j}$$

$$\frac{1.06}{1+i} = \frac{1}{1+j}$$

$$1+j = \frac{1+i}{1.06}$$

$$j = \frac{1+i}{1.06} - 1$$

$$j = \frac{1+i-1.06}{1.06} = \frac{i-.06}{1.06}$$

Immediate n pmts of R is 1^{st} pmt $k\%$ = geometric growth/decay

$(1+k)$ geometric factor.

eff. Rate is $i\%$

$$PV = \frac{R}{1+K} \, a_{\overline{n}|j} \qquad \text{where} \quad j = \frac{1+i}{1+K} - 1 = \frac{i-K}{1+K}$$

$$FV = PV(1+i)^n = \frac{R(1+i)^n}{1+K} \, a_{\overline{n}|j}$$

Future value of an annuity with payment varying in geometric progressions with common ratio $(1+k)$ is

$$FV = \frac{R}{1+k}(1+i)^n a_{\overline{n}|j}$$

$$j = \frac{1+i}{1+k} - 1 = \frac{i-k}{1+k}$$

or

$$FV = R\,\frac{(1+i)^n - (1+k)^n}{i-k}$$

Present value of a perpetuity-immediate with payments 1, 1(1+k), $1(1+k)^2$,... is

n pmts PV

$$PV = R\left[\frac{1-\left(\frac{1+k}{1+i}\right)^n}{i-k}\right]$$

$$PV = \frac{R}{1+k} a_{\overline{n}|j}$$

perpetuity

$\boxed{k < i}$

$$PV = \frac{R}{i-k}$$

$$PV = \frac{R}{1+k}\cdot\frac{1}{j}$$

Formulas for annuity-due and perpetuity-due can be obtained by multiplying the previous formulas by $1+i$.

$1 initial pmt

$$1+j = \frac{1+i}{1+K}$$

$$PV = \frac{1}{1+K} \cdot a_{\overline{n}|j} (1+i) = \frac{1+i}{1+K} a_{\overline{n}|j}$$

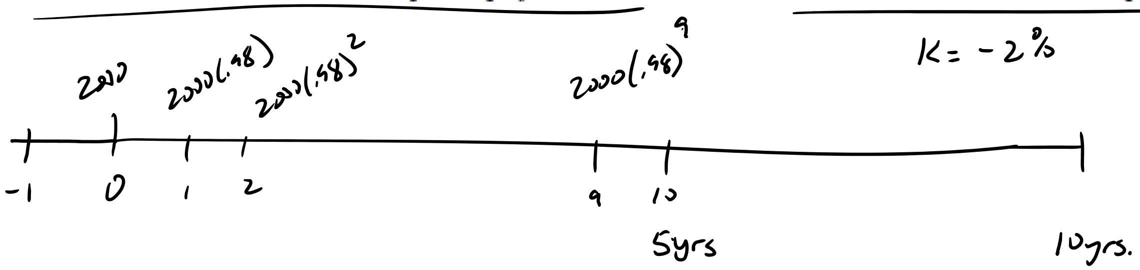
$$= (1+j) a_{\overline{n}|j} = \ddot{a}_{\overline{n}|j} = PV$$

$$PV = \frac{1 - \left(\frac{1+K}{1+i}\right)^n}{i-K} (1+i)$$

Perpetuity Due.

$$PV = \frac{1}{1+K} \cdot \frac{1}{j} (1+i) \quad \text{or} \quad PV = \frac{1}{i-K} \cdot (1+i)$$

Example: Find the accumulated value at the end of 10 years for an annuity in which payments are made at the beginning of each half year for 5 years. The first payment is 2000 and each payment thereafter is 98% of the prior payment. the interest rate is 10% convertible quarterly.



$K = -2\%$

$i^{(4)} = 10\%$

need $\frac{i^{(2)}}{2}$

with calc.

$i^{(2)} = 10.125\%$

$\frac{i^{(2)}}{2} = 5.0625\%$

Change to a fixed annuity

$$1+j = \frac{1+i}{1+K}$$

$$j = \frac{i-K}{1+K} = \frac{.050625 - -.02}{.98} = 0.07206632\,65$$

$$PV = 2000\,\ddot{a}_{\overline{10}|j} = 14916.69$$

$$FV = PV\left(1+\frac{.10}{4}\right)^{40} \quad \text{or} \quad FV = PV\left(1+\frac{.10125}{2}\right)^{20}$$

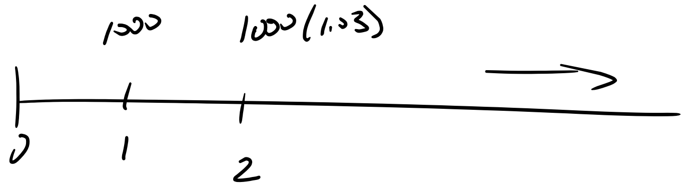
$$FV = \$40{,}046.89$$

method 2

$$FV = 2000\left[\frac{\left(1+\frac{i^{(2)}}{2}\right)^{10} - (.98)^{10}}{\frac{i^{(2)}}{2} - -.02}\right]\left(1+\frac{i^{(2)}}{2}\right)^{1}\cdot\left(1+\frac{i^{(2)}}{2}\right)^{10}$$

$K < i$

Example: Find the present value of a perpetuity-immediate if payments are made quarterly, the first payment made is \$1000 and each payment increases by 3%. The effective interest rate per quarter is 4%.



$K = 3\%$

$\frac{i^{(4)}}{4} = 4\%$

Quarterly eff. Rate.

$$PV = \frac{1000}{.04 - .03} = \frac{1000}{.01} = 100{,}000$$

In 4.6/4.7 the payment period and the interest conversion period are equal and coincide. If this is not the case, find the rate of interest convertible at the same frequency as payments are made.

Example: Find the present value of an annuity at the time of the first payment, where the first payment was \$15,000 and each subsequent payment decreases by \$2.000 until the last payment is \$1,000. The payments are made every other year and the annual effective interest rate is 6%.

$i = 6\%$

$j = $ 2yr eff.

8 pmts.

$P = 15000$

$Q = -2000$

$n = 8$

$1 + j = (1+i)^2$

$j = (1.06)^2 - 1$

$= .1236$

$$PV = P a_{\overline{n}|} + Q \frac{a_{\overline{n}|} - nv^n}{j} \quad \text{(Immediate)}$$

Due

$$PV = P \ddot{a}_{\overline{n}|} + Q \frac{a_{\overline{n}|} - nv^n}{j} (1+j)$$

$$PV = 15000 \, a_{\overline{8}|j} (1+j) + (-2000) \frac{a_{\overline{8}|j} - 8v^8}{j} (1+j)$$

$$= 82681.92 + -31937.14$$

$$= 50744.78$$

6. An insurance company has an obligation to pay the medical costs for a claimant. Average annual claims costs today are 6750, and medical inflation is expected to be 3.25% per year. The claimant is expected to live an additional 16 years. Claim payments are made at yearly intervals, with the first claim payment to be made one year from today. Find the present value of the obligation if the annual interest rate is 5.8%. **Answer: $88,329.18**

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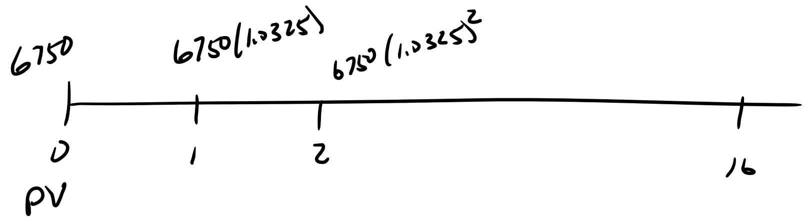

$i = 5.8\%$

$k = 3.25\%$

$n = 16$

$$PV = 6750(1.0325)\left[\frac{1 - \left(\frac{1.0325}{1.058}\right)^{16}}{.058 - .0325}\right]$$