Section 4.2: Different Payment and Interest Conversion Periods

The first approach is applicable if the only objective is to compute the numerical value of an annuity.

(1) Find the rate of interest, convertible at the same frequency as payments are made, that is equivalent to the given rate of interest.

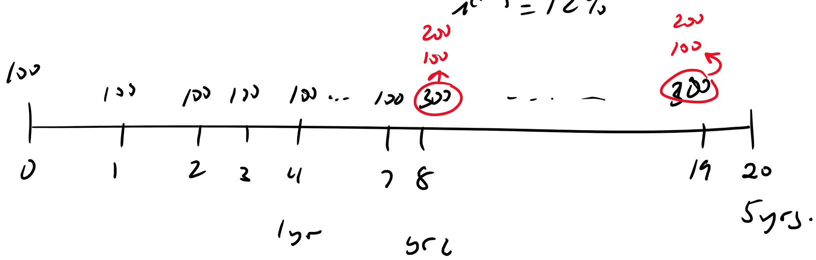
(2) Use this new rate of interest, find the value of the annuity using the techniques in chapter 3.

The second approach involves an algebraic analysis of the annuity.

Example: Find the accumulated value at the end of five years of an investment in which \$100 is deposited at the beginning of each quarter for the first two years and \$300 is deposited at the beginning of each quarter for the next three years, if the fund earns 12% convertible monthly.

need quarterly Rate per period

$i^{(12)} = 12\%$



$i^{(12)} = 12\% \to i^{(4)} = 12.1204\%$

$j = \frac{i^{(4)}}{4} = 3.0301\%$

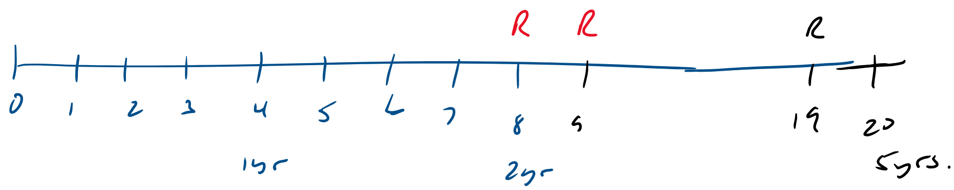
$d = \frac{j}{1+j}$ quarterly eff discount Rate

$$100\ddot{s}_{\overline{20}|j} + 200\ddot{s}_{\overline{12}|j} = \$5706.37$$

$$\left(100 s_{\overline{20}|j} + 200 s_{\overline{12}|j}\right)\left(1 + \frac{i^{(4)}}{4}\right) =$$

Example: Jason borrows $1500. Two years after the start of the loan, Jason will repay the loan with level payments quarterly for three years. If the rate of interest charged on the loan is 10% convertible semiannually, find the amount of each quarterly payment.

$$\underbrace{i^{(2)} = 10\%}$$
$$\rightarrow \frac{i^{(4)}}{4} = 2.469\%$$



$$PV = 1500 = R\, a_{\overline{12}|\frac{i^{(4)}}{4}} \cdot v^7$$

$$R = \frac{1500}{a_{\overline{12}|\frac{i^{(4)}}{4}} \cdot v^7} = \$173.14$$

alternate method

$$1500 = R\, \ddot{a}_{\overline{12}|\frac{i^{(4)}}{4}} \underbrace{\left(1 + \frac{i^{(2)}}{2}\right)^{-4}}_{v^8}$$

Section 4.3: Annuities Payable less Frequently than interest Convertible

Reminder: $\underbrace{1 + r + r^2 + r^3 + ... + r^{n-1}}$ $= \dfrac{1 - r^n}{1 - r}$

pmt period is annual.
Interest period is monthly.

Let k be the number of interest conversions in one payment period.

Let n be the term of the annuity measured in interest conversion periods.

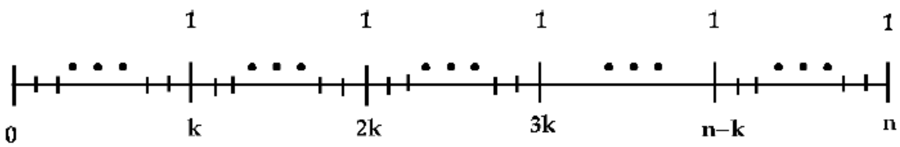
Let i be the rate of interest per interest conversion period. (an effective Rate per period)

Assume that each payment period contains an integral number of interest conversions periods; thus k and n are both positive integers. The number of payments made is $m = n/k$ is also a positive integer.

$n = mk$

Annuity-immediate

Find the present value of an annuity which pays 1 at the end of each k interest conversion periods for a total of n interest conversion periods.



$$PV = 1v^k + 1v^{2k} + 1v^{3k} + \cdots + 1v^{n-k} + v^n \qquad n = mk$$

$$= v^k + (v^k)^2 + (v^k)^3 + \cdots + (v^k)^{m-1} + (v^k)^m$$

$$= v^k\left[1 + v^k + (v^k)^2 + \cdots + (v^k)^{m-1}\right]$$

$$= v^k\left[\frac{1-(v^k)^m}{1-v^k}\right] = v^k \cdot \left(\frac{1-v^n}{1-v^k}\right)$$

$$= \frac{1}{(1+i)^k} \cdot \frac{1-v^n}{(1-v^k)} = \frac{1-v^n}{(1+i)^k - 1}$$

$$PV = \frac{\frac{1-v^n}{i}}{\frac{(1+i)^k-1}{i}} = \frac{a_{\overline{n}|i}}{s_{\overline{k}|i}} \qquad \frac{a_{\overline{mk}|i}}{s_{\overline{k}|i}}$$

Annuity Immediate

$$PV = \frac{a_{\overline{n}|i}}{s_{\overline{k}|i}}$$

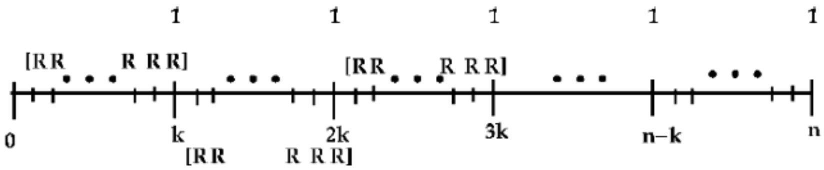
also can do. $\dfrac{\ddot{a}_{\overline{n}|i}}{\ddot{s}_{\overline{k}|i}}$ since $\dfrac{\frac{1-v^n}{d}}{\frac{(1+i)^k-1}{d}} = \dfrac{1-v^n}{(1+i)^k-1}$

$$FV = PV(1+i)^n = \underline{a_{\overline{n}|i}(1+i)^n} = \underline{s_{\overline{n}|i}}$$

$$FV = PV(1+i)^n = \frac{a_{\overline{n}|i}\,(1+i)^n}{s_{\overline{k}|i}} = \frac{s_{\overline{n}|i}}{s_{\overline{k}|i}}$$

Page 3: alternate derivation

annuity Immediate



Suppose we have a payment of R (end of each period) Such that

$$\underbrace{R\, s_{\overline{k}|i} = 1}$$

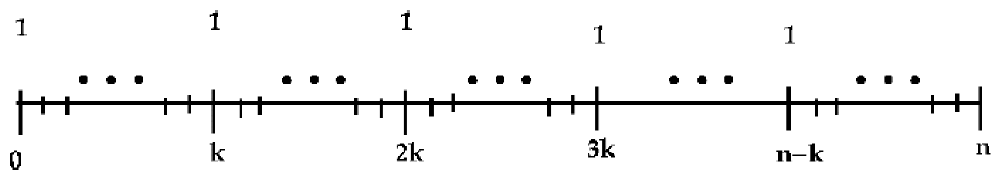
$$R = \frac{1}{s_{\overline{k}|i}}$$

$$PV = R\, a_{\overline{n}|i}$$

$$PV = \frac{1}{s_{\overline{k}|i}}\, a_{\overline{n}|i} = \frac{a_{\overline{n}|i}}{s_{\overline{k}|i}}$$

Annuity-due

Find the present value of an annuity which pays 1 at the beginning of each k interest conversion periods for a total of n interest conversion periods.



$pv = 1 + v^k + v^{2k} + v^{3k} + ... + v^{n-k}$

once again consider $n = km$ where m = the number of payments

$pv = 1 + v^k + v^{2k} + v^{3k} + ... + v^{km-k}$

$$pv = 1 + (v^k) + (v^k)^2 + (v^k)^3 + ... + (v^k)^{m-1} = \frac{1-(v^k)^m}{1-v^k} = \frac{1-v^n}{1-v^k} = \frac{\frac{1-v^n}{i}}{\frac{1-v^k}{i}} = \frac{a_{\overline{n}|i}}{a_{\overline{k}|i}}$$

$$fv = \frac{a_{\overline{n}|}}{a_{\overline{k}|}}(1+i)^n = \frac{s_{\overline{n}|}}{a_{\overline{k}|}}$$

OR $pv = \dfrac{\ddot{a}_{\overline{n}|i}}{\ddot{a}_{\overline{k}|i}}$

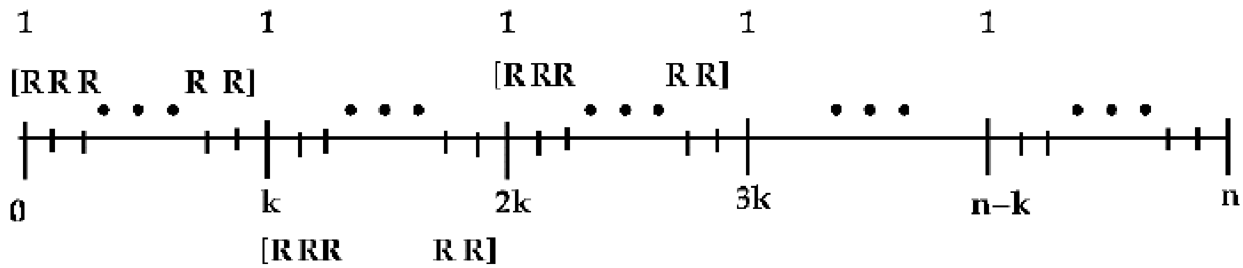
Page 5: alt def & perpetuity

alternate derivation.

Assume payments of R at every conversion period.

$\text{Pv} = \text{R}\ddot{a}_{\overline{n}|}$ where $1 = \text{R}\ddot{a}_{\overline{k}|}$

$$\text{thus } pv = \frac{\ddot{a}_{\overline{n}|}}{\ddot{a}_{\overline{k}|}} = \frac{\frac{1-v^n}{d}}{\frac{1-v^k}{d}} = \frac{1-v^n}{1-v^k} = \frac{a_{\overline{n}|}}{a_{\overline{k}|}}$$



Perpetuity

$$\text{immediate: } pv = \lim_{n\to\infty} \frac{a_{\overline{n}|}}{s_{\overline{k}|}} = \frac{1}{i\ s_{\overline{k}|}}$$

$$\frac{a_{\overline{\infty}|}}{s_{\overline{k}|}} = \frac{\frac{1}{i}}{s_{\overline{k}|}} = \frac{1}{i\ s_{\overline{k}|}}$$

$$\text{due: } pv = \frac{1}{i\ a_{\overline{k}|}} \text{ or } pv = \frac{1}{d\ \ddot{a}_{\overline{k}|}}$$

$$pv = \frac{\ddot{a}_{\overline{n}|}}{\ddot{a}_{\overline{k}|}}$$

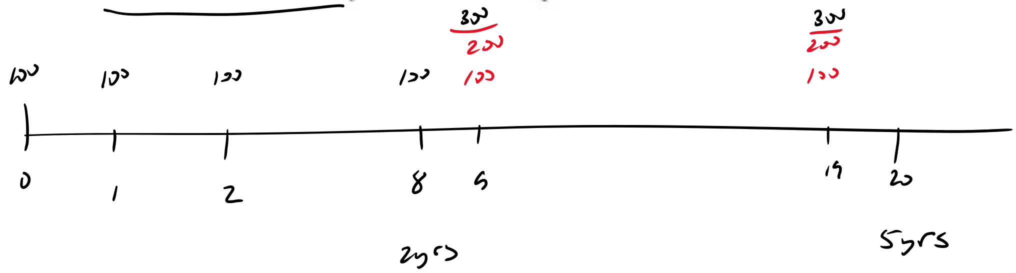
Example: Find the accumulated value at the end of five years of an investment in which \$100 is deposited at the beginning of each quarter for the first two years and \$300 is deposited at the beginning of each quarter for the next three years, if the fund earns 12% convertible monthly. Use the techniques from section 4.3.

quarterly pmts.

$i^{(12)} = 12\%$ nominal Rate.

each pmt period has 3 interest periods

$K = 3$



for \$100 pmts

$m = 20$

$n = mK = 60$

for \$200 pmts

$m = 12$

$n = 12(3) = 36$

monthly eff. Rate

$$\frac{i^{(12)}}{12} = \frac{12\%}{12} = 1\%$$

$$FV = \frac{100\, s_{\overline{60}|.01}}{a_{\overline{3}|.01}} + 200\,\frac{s_{\overline{36}|.01}}{a_{\overline{3}|.01}} = 5706.37$$

Section 4.4: Annuities payable more frequently than interest is convertible

Assume that the quoted interest rate is an effective annual rate of interest and the payments are made more frequently than once per year. This give the concept of a m-thly payable annuity.

Let i be the rate per interest conversion period (currently annually).

Let m be the number of payments in one interest conversion period.

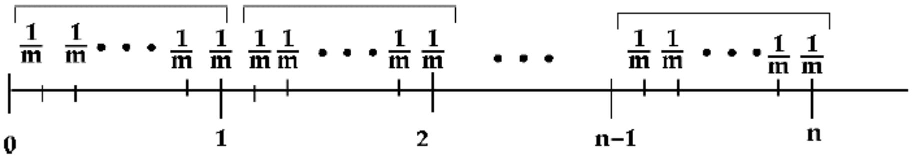
Let n be the term of the annuity measured in interest conversion periods. There are nm total payments made.

$nm =$ total # of pmts.

Present value of m-thly payable annuity-immediate

total pmt of 1 each yr.

Assume total annual payments of 1 payable m-thly with an annual effective rate i. Compute the present value. **Note:** If the total annual payment is 1 then the payment per payment period is $\frac{1}{m}$



$$pv = \frac{1}{m}v^{\frac{1}{m}} + \frac{1}{m}v^{\frac{2}{m}} + \frac{1}{m}v^{\frac{3}{m}} + \ldots + \frac{1}{m}v^{n-\frac{1}{m}} + \frac{1}{m}v^{n}$$

$$n - \frac{1}{m} = \frac{nm-1}{m}$$

$$pv = \frac{1}{m}v^{\frac{1}{m}}\left(1 + v^{\frac{1}{m}} + v^{\frac{2}{m}} + \ldots + v^{n-\frac{2}{m}} + v^{n-\frac{1}{m}}\right)$$

$$pv = \frac{1}{m}v^{\frac{1}{m}}\left(1 + v^{\frac{1}{m}} + \left(v^{\frac{1}{m}}\right)^2 + \ldots + \left(v^{\frac{1}{m}}\right)^{nm-2} + \left(v^{\frac{1}{m}}\right)^{nm-1}\right) = \frac{1}{m}v^{\frac{1}{m}}\left(\frac{1-\left(v^{\frac{1}{m}}\right)^{nm}}{1-v^{\frac{1}{m}}}\right)$$

$$pv = \frac{1-v^n}{m(1+i)^{\frac{1}{m}}(1-v^{\frac{1}{m}})} = \frac{1-v^n}{m\left[(1+i)^{\frac{1}{m}}-1\right]} = \frac{1-v^n}{i^{(m)}}$$

$$PV = a^{(m)}_{\overline{n}|i}$$

$$FV = a^{(m)}_{\overline{n}|i}(1+i)^n = \frac{1-v^n}{i^{(m)}}(1+i)^n$$

$$= \frac{(1+i)^n - 1}{i^{(m)}} = s^{(m)}_{\overline{n}|}$$

Ch 3 method.

$$800\, a_{\overline{20}|\, \frac{i^{(4)}}{4}} = PV.$$

Example: Ray secures a loan that requires him to pay $800 at the end of each quarter for 5 years. If the annual effective rate of interest is 10.25%, what is the amount of the loan?

$\boxed{\text{4.4 methods}}$

$i = 10.25\%$

Total Annual pmt

$R = 800(4) = 3200$

$$PV = 3200\, a^{(4)}_{\overline{5}|\,10.25\%}$$

use BA to set

$i^{(4)} = 9.878\%$

$$= 3200\ \frac{1-(1.1025)^{-5}}{i^{(4)}}$$

$$= 3200\ \frac{1-(1.1025)^{-5}}{.09878} = \$12{,}507.37$$

Other formulas for m-thly payable annuities.

$$s_{\overline{n}|i}^{(m)} = a_{\overline{n}|i}^{(m)}(1+i)^n = \frac{1-v^n}{i^{(m)}}(1+i)^n = \frac{(1+i)^n - 1}{i^{(m)}}$$

$$\ddot{a}_{\overline{n}|i}^{(m)} = \frac{1-v^n}{d^{(m)}}$$

$$\ddot{s}_{\overline{n}|i}^{(m)} = \frac{(1+i)^n - 1}{d^{(m)}}$$

$$a_{\overline{\infty}|i}^{(m)} = \frac{1}{i^{(m)}}$$

$$\ddot{a}_{\overline{\infty}|i}^{(m)} = \frac{1}{d^{(m)}}$$

Example: At what annual effective rate of interest is the present value of a series of payments of \$50 every other month forever, with the first payment made immediately, equal to \$1200?

perpetuity due.

find i (annual eff.)

Total annual pmt = $50(6) = 300$

$$1200 = 300\,\ddot{a}^{(6)}_{\overline{\infty}|}$$

$$1200 = \frac{300}{d^{(6)}}$$

$$d^{(6)} = \frac{300}{1200} = \frac{1}{4}$$

$$1+i = \left(1 - \frac{d^{(6)}}{6}\right)^{-6}$$

$$i = \left(1 - \frac{1}{4}\right)^{-6} - 1$$

$$= 29.0923\,\%$$

Ch 3 method

$$1200 = 50\,\ddot{a}_{\overline{\infty}|} = \frac{50}{d}$$

$$d = \underbrace{\frac{50}{1200} = \frac{d^{(6)}}{6}}$$

$$d^{(6)} = \frac{6(50)}{1200} = \frac{300}{1200} = \frac{1}{4}$$