Section 2.2: The Basic Problem

An Interest problem involves four basic quantities:

- The principal originally invested.
- The length of the investment period.
- The rate (or force) of interest (or discount).
- The accumulated value of the principal at the end of the investment period.

Observations:

- The length of the investment period is measured in time units, usually one year. However, if nominal rates of interest/discounts are involved the most convenient time unit is generally the interest conversion period.
- At this point, the phrase **yield rate** is that rate of interest which will establish an equivalency of value between a financial value at one point in time and a financial value at a different point in time.

  i.e. At time $t = 0$ \$100 is equivalent in value to \$110 at time $t = 1$, if and only if the yield rate is equal to 10%.
- An interest problem can be viewed from two perspectives: the borrower and the lender.
- In practical applications, the terminology can become confusing and inconsistent. Be sure to completely understand the exact nature of the transaction to determine what method to apply.

Section 2.3: Equations of Value

A fundamental principal in the theory of interest is that the value of an amount of money at any given point in time depends upon the time elapsed since the money was paid in the past or upon the time which will elapse in the future before it is paid.

Thus two or more amounts of money payable at different points in time can not be compared until all amounts are accumulated or discounted to a common date, the **comparison date**.

The equation which accumulates or discounts each payment to the comparison date is called the **equation of value**.

- For compound interest, the choice of the comparison date makes no difference in the answer obtained. There is a different equation of value for each comparison date.
- For simple interest or simple discount the choice of a comparison date does affect the answered obtained.

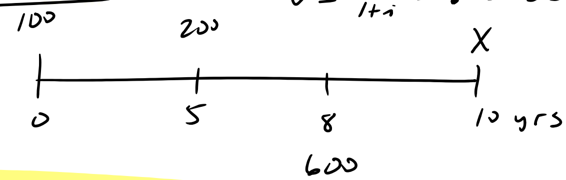
$i^{(2)} = 8\%$ used calc

annual eff. Rate $= 8.16\%$

$v = \frac{1}{1+i} = .924556$

Example: In return for a promise to receive \$600 at the end of 8 years, a person agrees to pay \$100 at once, \$200 at the end of 5 years, and to make a further payment at the end of 10 years. Find the payment at the end of 10 years if the nominal rate of interest is 8% convertible semiannually.

(a) Solve the problem using the comparison date at time 0.



time in yrs.

$$100 + 200v^5 + Xv^{10} = 600v^8$$

$$X = \frac{600v - 100 - 200v^5}{v^{10}} = 186.75$$

time is semiannual.

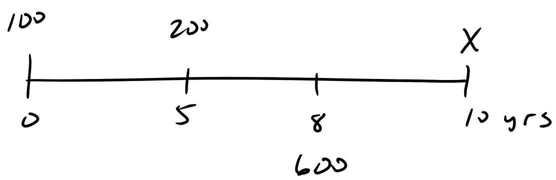
$i^{(2)} = 8\%$ $\quad i = \frac{i^{(2)}}{2} = 4\%$ semiannual eff. Rate.

$$V = \frac{1}{1+i} = \frac{1}{1.04}$$

$$100 + 200v^{10} + Xv^{20} = 600v^{16}$$

$$X = 186.75$$

(b) Solve using the equation of value at time 10.



yearly $\quad i =$ annual eff. Rate

$$100(1+i)^{10} + 200(1+i)^5 + X = 600(1+i)^2$$

$$\vdots$$

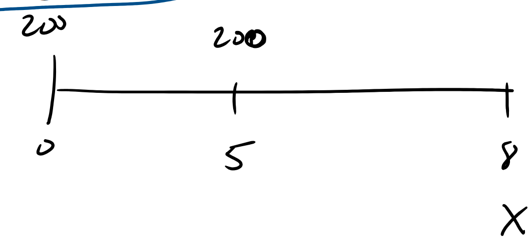
$$X = 186.75$$

$$100\left(1+\frac{i^{(2)}}{2}\right)^{20} + 200\left(1+\frac{i^{(2)}}{2}\right)^{10} + X = 600\left(1+\frac{i^{(2)}}{2}\right)^4$$

$$\frac{i^{(2)}}{2} = 4\%$$

Example: Find the amount to be paid at the end of 8 years which is equivalent to two payments of \$200 each, the first to be paid immediately and the second to be paid at the end of 5 years. Assume 7% simple interest is earned from the date of each payment is made and use a comparison date of

← annual simple interest Rate.



1. The end of 8 years.

$$200(1+.07(8)) + 200(1+.07(3)) = X$$

$$X = 554$$

2. The end of 12 years.

$$200(1+.07(12)) + 200(1+.07(7)) = X(1+.07(4))$$

$$\vdots$$

$$\underline{X = 520.31}$$

Section 2.4: unknown time

Example: Find the length of time necessary for \$500 to accumulate to \$2000 if invested at a nominal rate of 7% compounded quarterly.

$$500\left(1+\frac{.07}{4}\right)^{4t} = 2000$$

$$\left(1+\frac{.07}{4}\right)^{4t} = 4$$

$$4t \ln\left(1+\frac{.07}{4}\right) = \ln(4)$$

$$t = \frac{\ln(4)}{4\ln\left(1+\frac{.07}{4}\right)} = 19.9769 \text{ yrs.}$$

TVM Solver

N = 4t Solve

I% = 7/4

PV = -500

PMT = 0

FV = 2000

P/Y = C/Y = 1

n = 79.907963

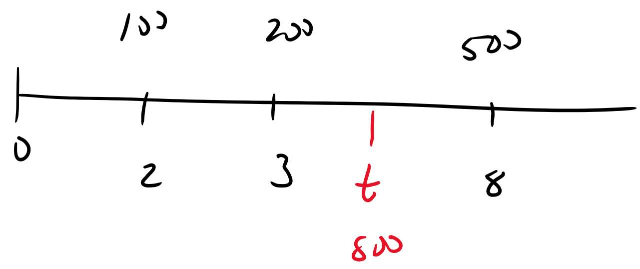
4t = n

t = 19.9769 yrs.

Example: Payments of \$100, \$200, and \$500 are due at the end of years 2, 3, and 8 respectively. Assuming an effective rate of interest of 5% per annum, find the point in time at which a payment of \$800 would be equivalent.

$i = 5\%$

$v = \frac{1}{1+i} = \frac{1}{1.05}$



time zero

$$100v^2 + 200v^3 + 500v^8 = 800v^t$$

$$A = \frac{100v^2 + 200v^3 + 500v^8}{800} = v^t$$

$$\ln A = t \ln v$$

$$t = \frac{\ln A}{\ln(v)} = 5.8318 \text{ yrs.}$$

Let amounts s_1, s_2, ..., s_n be paid at times t_1, t_2, ..., t_n respectively. Find a time t such that the sum of the payments paid at time t is equivalent to the payments made separately.

This can be approximated by using a weighted average of the various time payments, denoted by $\bar{t}$, is called the **method of equated time.**

$$\bar{t} = \frac{s_1 t_1 + s_2 t_2 + \cdots + s_n t_n}{s_1 + s_2 + \cdots + s_n} = \frac{\sum_{k=1}^{n} s_k t_k}{\sum_{k=1}^{n} s_k}$$

Example: Payments of \$100, \$200, and \$500 are due at the end of years 2, 3, and 8 respectively. Assuming an effective rate of interest of 5% per annum, find the point in time at which a payment of \$800 would be equivalent using the method of equated time. i.e. find $\bar{t}$.

$$\bar{t} = \frac{100(2) + 200(3) + 500(8)}{800} = 6 \text{ yrs.}$$

$\bar{t}$ is allways greater than the actual.

Only If you are interested. The *Rule of 72*, $t = 72/i\%$, approximates the time it takes for money to double at a given interest rate. Discussed on page 56-57.

Rate of Interest	Rule of 72	Exact value
4%	18	17.67
6	12	11.90
8	9	9.01
10	7.2	7.27
12	6	6.12
18	4	4.19

Section 2.5: Unknown Rate of Interest

Example: Find the nominal rate of interest compounded monthly so that the accumulated value of 400 will be 600 in 4 years.

$$400\left(1+\frac{i^{(12)}}{12}\right)^{48} = 600$$

$$\left(1+\frac{i^{(12)}}{12}\right)^{48} = \frac{600}{400}$$

$$1+\frac{i^{(12)}}{12} = \left(\frac{6}{4}\right)^{1/48}$$

$$i^{(12)} = 12\left\{\left(\frac{6}{4}\right)^{1/48} - 1\right\} = 10.17956\%$$

TVM solver

N = 48

I% = monthly eff. ← solve

PV = -400

PMT = 0

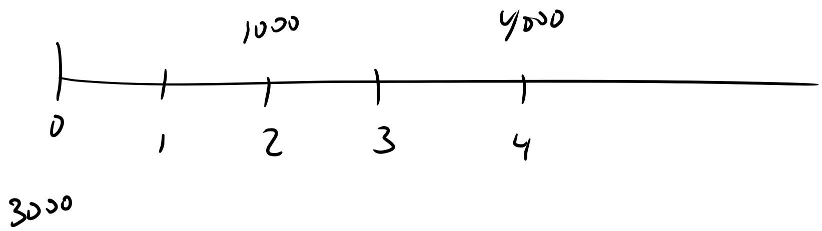
FV = 600

P/Y = C/Y = 1

$$\frac{i^{(12)}}{12} = .848296$$

$$i^{(12)} = 10.17956$$

Example: At what effective rate of interest will the present value of \$1000 at the end of two years and \$4000 at the end of 4 years be equal to \$3000?



$$1000v^2 + 4000v^4 = 3000$$

$$4v^4 + v^2 - 3 = 0$$

by quadratic formula

$$v^2 = \frac{-1 \pm \sqrt{1 - 4(4)(-3)}}{2(4)}$$

$$v^2 = .75 \qquad v^2 = -1$$

$$v = \frac{1}{1+i}$$

$$(1+i)^2 = (.75)^{-1}$$

$$\vdots$$

$$i = 15.47\%$$

Cash flow

BA)

$CF0 = -3000$

$C01 = 0$

$F01 = 1$

$C02 = 1000$

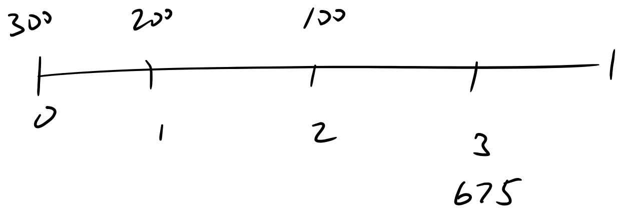
$F02 = 1$

$C03 = 0$

$F03 = 1$

$C04 = 4000$

Example: Find an expression for the exact effective rate of interest at which payments of \$300 at the present, \$200 at the end of one year, and \$100 at the end of 2 years will accumulate to \$675 at the end of 3 years.



$$300(1+i)^3 + 200(1+i)^2 + 100(1+i) = 675$$

Let $x = 1+i$ $\quad 300x^3 + 200x^2 + 100x = 675$

Solve for $x = 1+i$

$i = 5.146\%$

Cash flow option on the calc.

BA)

CF0 = 300

C01 = 200

F01 = 1

C02 = 100

F02 = 1

C03 = -675

F03 = 1

IRR = compute.

5.146%

- Internal Rate of Return (IRR): The IRR is that rate of interest at which the present value of net cash flows *from* the investment is equal to the present value of net cash flows *into* the investment. In other words, it's the value of i such that Net Present Value(NPV)= 0.
  - The IRR of an investment may or may not be unique (and may or may not be a real number).
  - The built-in IRR function in the BAII will return the IRR that is closest to 0 in the case that there are multiple IRR's.
- BAII
  - Press the CF button.
  - Clear the list by pressing 2nd and then CE—C
  - you must provide a cash flow and a frequency for all periods. Be sure to press the enter button.
  - To solve press the IRR button and then press CPT
- TI-84
  - choice 8 of the Finance menu under the APPS button.
  - IRR(Initial Outlay, {Cash Flows}, {Cash Flow Counts})
  - you must provide a cash flow and a frequency for all periods.

Section 2.6: Determining Time Periods

Three methods for counting days (usually used with simple interest).

Method 1: Use the exact number of days for the period and to use 365 days in a year. Computing this way is sometimes called **exact simple interest** and is often denoted by actual/actual. Most times leap year is counted as having 366 days (sometimes not).

- ti-84: dbd(date1, date 2)
  where date1 and date 2 are of the form either DDMM.YY or MM.DDYY
  must use same form for both dates.
- BA: 2nd DATE. Note the dates are in the form MM.DDYY
  This can be used for both method 1 and method 2.

Method 2: Assume each month has 30 days and the year has 360 days. Computing this way is called **ordinary simple interest** and is often denoted by 30/360.

$$360(Y_2 - Y_1) + 30(M_2 - M_1) + (D_2 - D_1)$$

Method 3: Use the exact numbers of days in the investment but uses 360 days in a year. This is called the **Banker's Rule** and is ofter denoted by actual/360. Note: This method is always more favorable to a lender than exact simple interest.

$FV = PV(1+it)$
$= PV + \underline{PVit}$

Unless otherwise stated, it is assumed that in counting days interest is not credited for both the date of deposit and the date of withdrawal, but for only one of these two dates.

Example: Find the amount of interest that \$1000 deposited on September 13, 2018 will earn if the money is withdrawn on May 5, 2019 if the rate of simple interest is 8% based on

(a) exact simple interest (actual/actual method).

dbd(09.1318, 05.0519)
234 days

$$\text{Interest} = 1000(.08)\frac{234}{365}$$
$$= 51.29$$

(b) Ordinary simple interest (30/360 method).

30/360
232 days.

$$\text{Interest} = 1000(.08)\frac{232}{360} = 51.56$$

(c) Banker's rule (actual/360).

$$\text{Interest} = 1000(.08)\frac{234}{360} = \$52$$

Section 2.7: Practical examples

"real life" applications of interest can be creative and interesting.

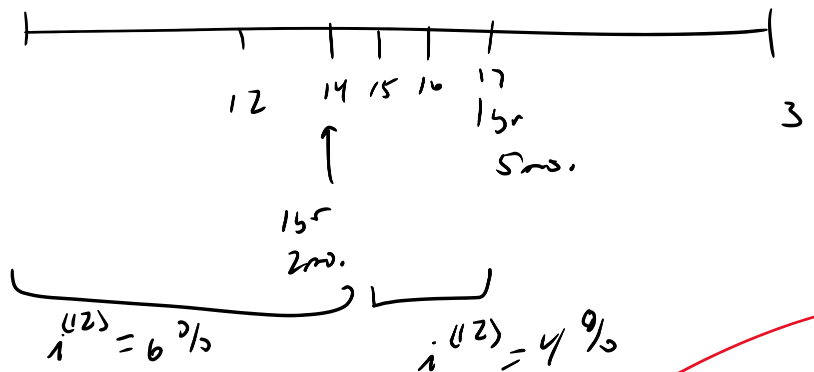
- Sometimes financial institutions will advertise two different rates on a deposit.

  For example, a certificate of deposit(CD) having a 4.55% rate/ 4.65% yield.

- On Us Treasury Bills (T-bills), the rates given for periods up to 1 year(13, 26, and 52 weeks) are actually rates of discount. For T-bills with longer term the rates are rates of interest.

- For accounts where the balance changes during the period, interest may be computed on average daily balance, minimum daily balance, beginning balance which is reduce by any withdrawals in the time period.

Example: You invest \$2000 in a three-year CD crediting 6% convertible monthly. If the CD is redeemed early, the credited rate will be reduced to 4% convertible monthly for the final three months of the period of investment. Find the amount you would receive if the CD is redeemed after 1 year 5 months.



$12\left(\frac{14}{12}\right)$

$12 \cdot \left(\frac{3}{12}\right)$

$$2000\left(1+\frac{.06}{12}\right)^{14}\left(1+\frac{.04}{12}\right)^{3} = \$2166.16$$