Section 11.7: Immunization

Redington immunization is another technique that has been developed to structure assets and liabilities in a manner that would reduce the effects of interest-rate fluctuations.

Redington immunization is more flexible than absolute matching since it does not require exact matching of an asset cash flow for each liability cash flow.

Assumption when using Redington Immunization

1) the yield curve is flat. i.e. spot rates s_i are equal for all i.

2) Interest rate changes are parallel shifts up or down.

Requirements for a Portfolio to be Redington immunized

1) The present value of the inflows from assets must equal the present value of the outflows.

net present value

$P_A(i) = P_L(i)$ or $P(i) = P_A(i) - P_L(i) = 0$

$P_A \geq P_L$ (some sources)

2) The modified duration of assets must equal the modified duration of liabilities.

$\bar{v}_A = \bar{v}_L$ or $P'(i) = 0$

$\frac{-P_A'(i)}{P_A(i)} = \frac{-P_L'(i)}{P_L(i)} \Rightarrow P_A'(i) = P_L'(i)$

$\bar{d}_A = \bar{d}_L$

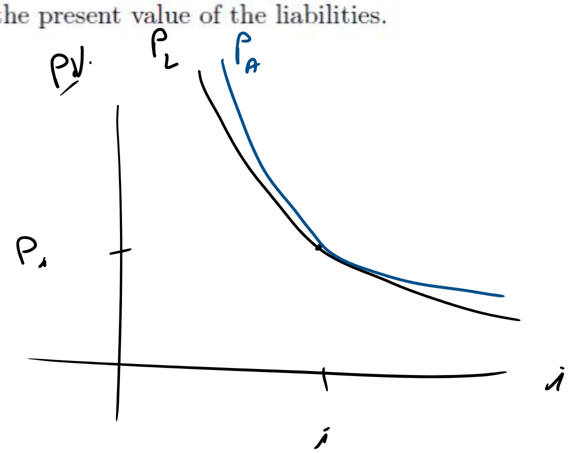
3) The convexity of the assets must be greater than the convexity of the liabilities.

$\bar{c}_A > \bar{c}_L$ or $P''(i) > 0$

$\frac{P_A''}{P_A} > \frac{P_L''}{P_A} \Rightarrow P_A'' > P_L''$

Condition 2 and 3 are so that the interest rate risk for the assets offsets the interest rate risk for the liability.

If the interest rate falls, the present value of the assets will increase by more than the present value of the liabilities.



If the immunization conditions are satisfied, then

$P_A(i) = P_L(i)$, $\overline{v}_A = \overline{v}_L$ $\overline{c}_A > \overline{c}_L$.

The approximation to the present value of assets and liabilities are:

$$P_A(i+h) \approx P_A(i)\left(1 - \overline{v}_A h + \frac{h^2}{2}\overline{c}_A\right) \qquad P_L(i+h) \approx P_L(i)\left(1 - \overline{v}_L h + \frac{h^2}{2}\overline{c}_L\right)$$

Thus

$$P_A(i+h) - P_L(i+h) \approx P_A(i)\left(1 - \overline{v}_A h + \frac{h^2}{2}\overline{c}_A\right) - P_L(i)\left(1 - \overline{v}_L h + \frac{h^2}{2}\overline{c}_L\right)$$

$$P_A(i+h) - P_L(i+h) \approx P_A(i)\left(1 - \overline{v}_A h + \frac{h^2}{2}\overline{c}_A - 1 + \overline{v}_L h - \frac{h^2}{2}\overline{c}_L\right)$$

$$P_A(i+h) - P_L(i+h) \approx P_A(i)\left(\frac{h^2}{2}\overline{c}_A - \frac{h^2}{2}\overline{c}_L\right) = P_A(i)\frac{h^2}{2}(\overline{c}_A - \overline{c}_L) > 0$$

Difficulties and Limitations of Redington immunization in practice

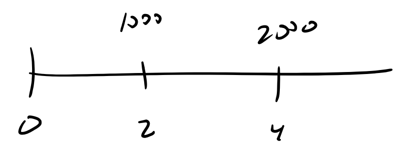
- The choice of one interest rate for all calculations.
- Only provides protection against small changes in i.
- Yield curves are usually not flat.
- Requires frequent rebalancing of portfolio to keep $\overline{v}_A = \overline{v}_L$
- Exact cash flows may not be known and may have to be estimated. i.e. callable bonds, prepaid mortgages.
- Assets may not exist in the right maturities to achieve immunization.

Example: A financial institution has to pay \$1,000 two years from now and \$2,000 four years from now. The current market interest rate is 10% and the yield curve is assumed to be flat at any time. The institution wishes to immunize the interest rate risk by purchasing zero coupon bonds which mature after 1, 3, and 5 years. A financial risk consultant suggests the following strategy:

- Purchase a 1-year zero coupon bond with a face value of \$154.16.
- Purchase a 3-year zero coupon bond with a face value of \$2,186.04.
- Purchase a 5-year zero coupon bond with a face value of \$660.18.

Show that this portfolio satisfies the three conditions of Redington immunization.

Liabilities



$$P_L = 1000v^2 + 2000v^4$$

$$P_L(.10) = 2192.47$$

equal ✓.

$$P_A = \frac{154.16}{(1+i)} + \frac{2186.04}{(1+i)^3} + \frac{660.18}{(1+i)^5}$$

$$P_A(10\%) = 2192.47$$

$$\bar{d}_A = \frac{\sum t R_t v^t}{P_A} = \frac{1(154.16)v^1 + 3(2186.04)v^3 + 5(660.18)v^5}{2192.47} = 3.2461$$

$$\bar{d}_L = \frac{2(1000)v^2 + 4(2000)v^4}{P_L} = 3.2461$$

$$\bar{d}_A = \bar{d}_L$$ ✓

$$\bar{C}_A = \frac{P_A''(i)}{P_A(i)} = \frac{1(2)(154.16)v^3 + 3(4)(2186.04)v^5 + 5(6)(660.18)v^7}{2192.47} = 12.1704$$

$$\bar{C}_A > \bar{C}_L$$

$$\bar{C}_L = \frac{P_L''(i)}{P_L(i)} = \frac{2(3)(1000)v^4 + 4(5)(2000)v^6}{2192.47} = 12.1676$$

conditions of Redington Immunization hold.

Example: Define surplus $S = P_A - P_L$. Calculate the company's surplus in the previous example if there is an immediate one-time change in interest rates from 10% to 9%

$P_A(9\%) = 2258.53$

$P_L(9\%) = 2258.53$

$S = 0$

i	P_A	P_L	S
9%	2258.53	2258.53	0
10%	2,192.47	2,192.47	0.00
11%	2,219.08	2,2129.08	0.00
15%	1,899.64	1,899.65	−0.02
30%	1,291.40	1,291.97	−0.57
80%	495.42	499.16	−3.74

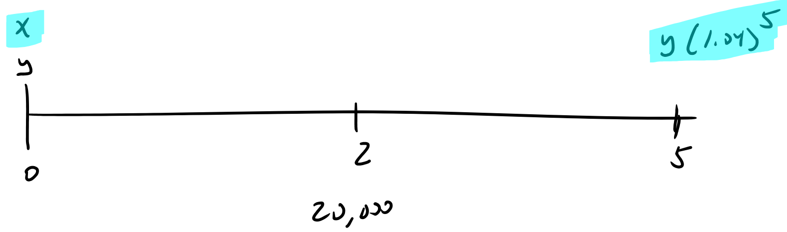
Example: An actuarial department needs to set-up an investment program to pay for a loan of \$20,000 due in 2 years. The only available investments are:

(i) a money market fund paying the current rate of interest.

(ii) 5-year zero-coupon bonds earning 4%.

Assume that the current rate of interest is 4%. Develop an investment program satisfying the theory of immunization. Graph the present value of asset minus liabilities versus interest rates.

Solution: Assume that x is invested in the money market fund and y in the zero coupon bonds.



$$P(i) = P_A - P_L = x + y(1.04)^5(1+i)^{-5} - 20000(1+i)^{-2}$$

$$P'(i) = -5y(1.04)^5(1+i)^{-6} + 40000(1+i)^{-3}$$

at $i = 4\%$ we need $P(4\%) = 0$ $P'(4\%) = 0$

$$P(4\%) = 0 = x + y(1.04)^5(1.04)^{-5} - 20000(1.04)^{-2}$$

$$\boxed{0 = x + y - \frac{20000}{(1.04)^2}}$$

$$P'(4\%) = 0 = -5y(1.04)^5(1.04)^{-6} + 40000(1.04)^{-3}$$

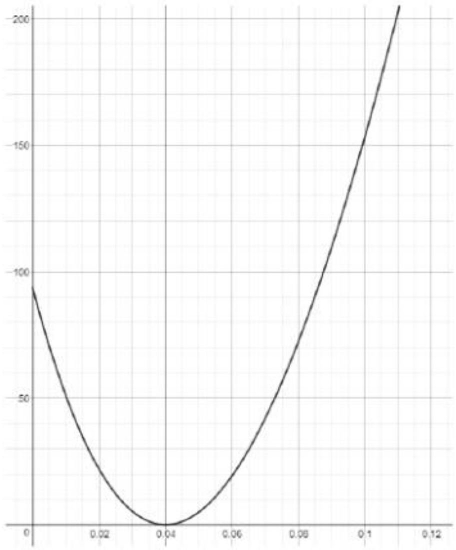
$$0 = -5y(1.04)^{-1} + \frac{40000}{(1.04)^3}$$

$$5y(1.04)^{-1} = \frac{40000}{(1.04)^3}$$

$$y = \frac{40000}{5(1.04)^2} = 7396.45$$

$x = \$11,094.67$

$p(i)$



check $\bar{C}_A > \bar{C}_L$ by $p''(i) > 0$ for $i = 4\%$

$$p''(i) = 30y\,(1.04)^5(1+i)^{-7} - 120000(1+i)^{-4}$$

$$p''(4\%) = 102576.5 > 0$$

Section 11.8: Full Immunization

Full immunization is achieved if $P_A \geq P_L$ for all $i > 0$.
i.e. $P(i) \geq 0$ for all $i > 0$.