Section 11.5: Analysis of Portfolios

Multiple Securities

An investment portfolio usually will contain multiple securities. Let P_j, $\overline{v}_j$, $\overline{c}_j$ be the price (or market value), modified duration, and convexity, respectively of the j-th security, $j = 1, 2, \cdots m$. Let P, $\overline{v}$, and $\overline{c}$ be the comparable values for the entire portfolio. Then $P = P_1 + P_2 + \cdots + P_m$

Modified Duration: $\overline{v} = \dfrac{-P'(i)}{P(i)} = \dfrac{-\frac{d}{di}[P]}{P(i)}$

$$= \frac{-[P'_1 + P'_2 + \cdots + P'_m]}{P(i)} = -\left[\frac{P'_1}{P(i)} + \frac{P'_2}{P(i)} + \cdots + \frac{P'_m}{P(i)}\right]$$

$$= -\left[\frac{P'_1}{P_1} \cdot \frac{P_1}{P(i)} + \frac{P'_2}{P_2} \cdot \frac{P_2}{P(i)} + \cdots + \frac{P'_m}{P_m} \cdot \frac{P_m}{P(i)}\right]$$

$$= \underbrace{-\frac{P'_1}{P_1}} \cdot \frac{P_1}{P(i)} + \underbrace{\frac{-P'_2}{P_2}} \cdot \frac{P_2}{P(i)} + \cdots + \underbrace{\frac{-P'_m}{P_m}} \cdot \frac{P_m}{P(i)}$$

$$= \overline{v}_1 \frac{P_1}{P(i)} + \overline{v}_2 \frac{P_2}{P(i)} + \cdots + \overline{v}_m \frac{P_m}{P(i)}$$

The modified duration of the entire portfolio is simply the weighted average of the modified durations of each security in the portfolio, where the weights are the fraction of the entire portfolio applicable to each security.

Macaulay Duration: $\overline{d} = \dfrac{\overline{d}_1 P_1 + \overline{d}_2 P_2 + \cdots + \overline{d}_m P_m}{P_1 + P_2 + \cdots + P_m} = \dfrac{\overline{d}_1 P_1 + \overline{d}_2 P_2 + \cdots + \overline{d}_m P_m}{P}$

Convexity: $\overline{c} = \dfrac{\overline{c}_1 P_1 + \overline{c}_2 P_2 + \cdots + \overline{c}_m P_m}{P_1 + P_2 + \cdots + P_m} = \dfrac{\overline{c}_1 P_1 + \overline{c}_2 P_2 + \cdots + \overline{c}_m P_m}{P}$

Example: A fund has a portfolio of investments with the following information.

	Present value	Macaulay's duration	convexity
Bond A	100,000	5.3	1.2
Bond B	50,000	3.4	3.2
Bond C	120,000	12.2	6.2
Bond D	80,000	2.3	3.6

Total = 350,000

Find the Macaulay's duration and the convexity for the entire portfolio.

$$\bar{d} = 5.3\left(\frac{100}{350}\right) + 3.4\left(\frac{50}{350}\right) + 12.2\left(\frac{120}{350}\right) + 2.3\left(\frac{80}{350}\right) = 6.70857 \text{ yrs}$$

$$\bar{C} = 1.2\left(\frac{100}{350}\right) + 3.2\left(\frac{50}{350}\right) + 6.2\left(\frac{120}{350}\right) + 3.6\left(\frac{80}{350}\right) = 3.7486$$

Example: An investment fund wants to invests \$100,000 in a mix of 5 year zero coupon bonds yielding 6% and 10-year zero coupon bonds yielding 7% such that the modified duration of the portfolio will equal 7. Find the amount that should be invested in each type of bond.

$\bar{v} = 7$

Let $P_1 = x$ Then $P_2 = 100000 - x$

P_1 is PV. of 5yr Bond.

P_2 is PV. of 10 yr Bond.

$$7 = \bar{v} = \bar{v}_1 \frac{P_1}{P} + \bar{v}_2 \frac{P_2}{P}$$

5yr Zero coupon Bond

$\bar{d} = 5$

$\bar{v} = (1.06)^{-1}(5) = \frac{5}{1.06}$

10yr Zero coupon Bond

$\bar{d} = 10$

$\bar{v} = (1.07)^{-1}(10) = \frac{10}{1.07}$

$$7 = \frac{5}{1.06} \frac{x}{100000} + \frac{10}{1.07} \frac{100000 - x}{100000}$$

$\vdots$

$x = \underline{50678.10}$ 5yr Bond

$100000 - x = \underline{49321.90}$ 10yr Bond

Duration and Passage of Time

In a portfolio, different securities have different cash flows payment dates. Thus the need to be able to calculate the duration of a security at any future date after issue. As expected the duration is calculated using a time-line and the remaining time-interval from the date of calculations to the dates of future payments.

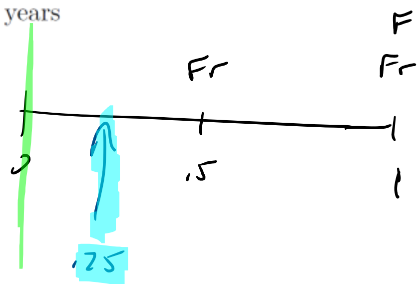
As time passes, the duration of a security with multiple future payments does decline. However at the time a cash flow occurs there is a discontinuity and the duration spikes upward. The duration is greater just after the cash flow than it was just before the cash flow. This in an effect of the weighted average of the times of the payments.

Example: A one year bond with 20% semiannual coupons has a nominal yield of 20% compounded semiannually. Calculate the Macaulay duration at quarterly intervals over the entire life of the bond.

coupon (.1)F   F + .1F = 1.1F

at $t = 0$:

$$\bar{d} = \frac{0.5(0.1)F\left(1+\frac{.2}{2}\right)^{-2*0.5} + 1 * (\overset{1.1}{\cancel{0.1}})F(1.1)^{-2}}{(0.1)F(1.1)^{-1} + 1.1F(1.1)^{-2}} = \frac{0.5(0.1)(1.1)^{-1} + 1 * (\overset{1.1}{\cancel{0.1}})(1.1)^{-2}}{(0.1)(1.1)^{-1} + 1.1(1.1)^{-2}} = 0.955 \text{ years}$$

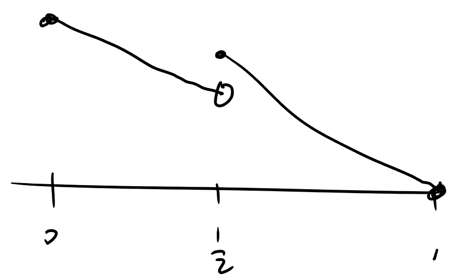


at $t = 0.25$:

$$\bar{d} = \frac{0.25(0.1)(1.1)^{-0.5} + 0.75 * (\overset{1.1}{\cancel{0.1}})(1.1)^{-1.5}}{(0.1)(1.1)^{-0.5} + 1.1(1.1)^{-1.5}} = 0.705 \text{ years}$$

t = .5 (just before the coupon)

$$\bar{d} \quad \frac{0(.1)(1.1)^0 + .5(1.1)(1.1)^{-1}}{(0.1)(1.1)^0 + 1.1(1.1)^{-1}} = .455 \text{ yrs.}$$



at $t = 0.5$(just after the coupon is paid):

$\bar{d} =$ .5

at $t = 0.75$:

$\bar{d} =$ .25

Section 1.6: Matching Assets and Liabilities

Now we consider a whole collection of transactions. specifically, the interrelationship between assets and liabilities for some financial enterprise, such as a bank, an insurance company, or a pension fund.

The assets will generate a series of cash inflows, A_t = cash inflow at time t.

The liabilities will generate a series of cash outflows, L_t = cash outflow at time t.

The goal is to achieve an equilibrium or safe balance between these cash inflows and outflows. If this equilibrium does not exist, then there is a risk of adverse effects created by changes in the level of interest rates.

Consider a bank that issues a 1-year cd with a guaranteed rate of interest. The bank will have assets backing this investment.

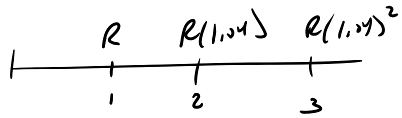
If the assets are invested "too long"(duration more than one), then the bank is vulnerable to loss if interest rates rise. They may have to sell assets to cover the contract redemptions. Due to the rising of the interest rates, the values of the assets have declined and thus losses may be incurred.

If the assets are invested "too short" (duration closer to zero), then the bank is vulnerable to loss if the interest rates fall. The interest earnings will decline and may not be sufficient to pay the guaranteed interest on the contracts at the end of the year.

We will look at three methods to address this issue: Absolute Matching/Dedication, Redington Immunization, and Full Immunization.

Absolute Matching/Dedication

Absolute Matching/Dedication is an approach to structure an asset portfolio in such a fashion that the cash inflow that will be generated will exactly match the cash outflow from the liabilities in every period. An example of this is a dedicated bond portfolio, usually composed of high-grade non-callable bonds.



Example: Several years ago a corporation entered into a 10-year lease agreement under which the corporation will make 10 annual lease payments. The first payment was \$840,000 and each payment hereafter was to be adjusted for inflation with annual 4% increases(compounded). The corporation has just made its 7th lease payment and has enough extra funds on hand that it wishes to "pre-fund" its three remaining lease payments. It decides to do so using an absolute matching strategy with zero coupon bonds. The current yield curve for zero coupon bonds shows the following rates. Find the amount that the corporation will need to implement this strategy.

Term	1 year	2 year	3 year
Spot rate	7.00%	8.00%	8.75%

8^{th} pmt $\quad 840000(1.04)^7 = 1{,}105{,}383 = A$

9^{th} pmt $\quad 840000(1.04)^8 = 1{,}149{,}598 = B$

10^{th} pmts $\quad 840000(1.04)^9 = 1{,}195{,}582 = C$

$$PV_A = \frac{A}{1.07} = 1{,}033{,}068.22$$

$$PV_B = \frac{B}{(1.08)^2} = 985{,}594.99$$

$$PV_C = \frac{C}{(1.0875)^3} = 929{,}590.27$$

Total

2,948,253.49

If constructing a dedicated bond portfolio with coupon bonds, we must select coupon bonds with maturity dates matching each of the cash outflow dates and also display the cash flow from each of the bonds on a time-line. We start at the end and match the last cash outflow with the longest term bond and work backwards recursively to the first cash outflow.

Example Rework the last example using coupon bonds with annual coupons. Assume that the coupon bonds all have the same yield rate as the zero coupon bonds and that all sell at par.

	Bond	Bond	Bond
Term	1 year	2 year	3 year
~~Spot rates~~ yield Rate	7.00%	8.00%	8.75%
Coupon Rate	7%	8%	8.75%

8th pmt $840000(1.04)^7 = 1,105,383 = A$

9th pmt $840000(1.04)^8 = 1,149,598 = B$

10th pmts $840000(1.04)^9 = 1,195,582 = C$

Liabilities	A	B	C
Bond 1	$x + .07x$ $1.07x$		
Bond 2	$.08y$	$1.08y$	
Bond 3	$.0875z$	$.0875z$	$1.0875z$

Redemption Amt $F = C$

Let x = face value of Bond 1

y = face value of Bond 2

z = face value of Bond 3

$1.0875z = C$

$$z = \frac{C}{1.0875} = \frac{1195582}{1.0875} = 1,099,386$$ face value of Bond 3

coupon for z $\quad 1099386(.0875) = 96,196.27$

$1.08y + .0875z = B \Rightarrow y = \frac{B - .0875z}{1.08} = 975,372$

$1.07x + .08y + .0875z = A \Rightarrow x = 870,241$

Spend $x + y + z$ since sold at par

Spend $x + y + z$ since sold at par

2,944,999

Example: A bond portfolio manager in a pension fund is designing a bond portfolio. His company has an obligation to pay 50000 at the end of each year for 3 years. He can purchase a combination of the following three bonds in order to exactly match its obligation:

- 1-year 5% annual coupon bond with a yield rate of 6%.
- 2-year 7% annual coupon bond with a yield rate of 7%.
- 3-year 9% annual coupon bond with a yield rate of 8%.

(A) How much of each bond should you purchase in order to exactly match the liabilities?

(B) Find the cost of such a combination of bonds

$x =$ face value of Bond 1 ie F

$y =$ face value of Bond 2

$z =$ face value of Bond 3

	yr 1	yr 2	yr 3
Liabilities	5000	5000	5000
Bond 1	$1.05x$		
Bond 2	$.07y$	$1.07y$	
Bond 3	$.09z$	$.09z$	$1.09z$

$$1.09z = 5000$$
$$1.07y + .09z = 5000$$
$$1.05x + .07y + .09z = 5000$$

$\longrightarrow$

A)

$z = 45872$

$y = 42871$

$x = 40829$

Price Bond 3: $45872(.09)\,a_{\overline{3}|8\%} + 45872(1.08)^{-3} = 47{,}054.16$

Bond 2: $42871(.07)\,a_{\overline{2}|7\%} + 42871(1.07)^{-2} = 42{,}871$

Bond 1: $40829(.05)\,a_{\overline{1}|6\%} + 40829(1.06)^{-1} = 40{,}443.578$

$130{,}369.14$