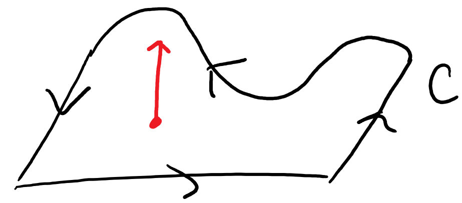
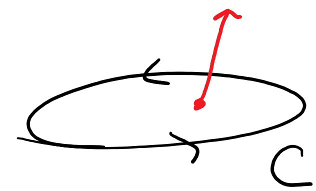
Section 16.8: Stokes' Theorem





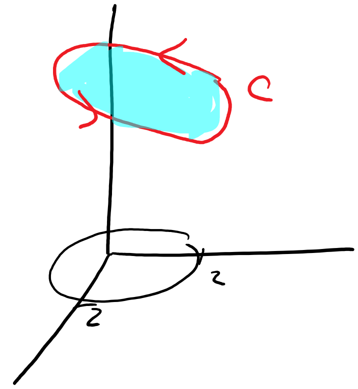
Stokes' Theorem: Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\Re^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_S \text{curl } \mathbf{F} \cdot (r_u \times r_v)\, dS$$

Curve C has **positive orientation** means if you walk the curve such that the surface is to your left, then your head is in the direction of the normal vector.

Positive orientation is determined by the right-hand rule. The normal vector points in the same direction as your right thumb if the fingers on your right hand are pointing in the direction the curve is traversed.

Example: Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle -y^2, x, z^2 \rangle$ and C is the curve of intersection of the plane $y + z = 3$ and the cylinder $x^2 + y^2 = 4$. Orient C to be counter clockwise when viewed from above.



$z = 3 - y$

$0 \le \theta \le 2\pi$

Method 1: parametrize the curve by $\mathbf{r}(\theta) = \langle 2\cos\theta, 2\sin\theta, 3 - 2\sin\theta \rangle$.

$\mathbf{F} = \langle -4\sin^2\theta, 2\cos\theta, (3 - \sin\theta)^2 \rangle$ and $\mathbf{r}'(\theta) = \langle -2\sin\theta, 2\cos\theta, -2\cos\theta \rangle$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}' d\theta = \int_0^{2\pi} \left(8\sin^3\theta + 4\cos^2\theta - 2\cos\theta(3 - \sin\theta)^2\right) d\theta$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$$

$$\text{curl}\,\mathbf{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$$

$$\begin{array}{cccc} & P & Q & R \\ \mathbf{F} = & \langle -y^2, & x, & z^2 \rangle \end{array}$$

$$\text{curl}\,\mathbf{F} = \langle 0 - 0,\ 0 - 0,\ 1 + 2y \rangle = \langle 0, 0, 1 + 2y \rangle$$

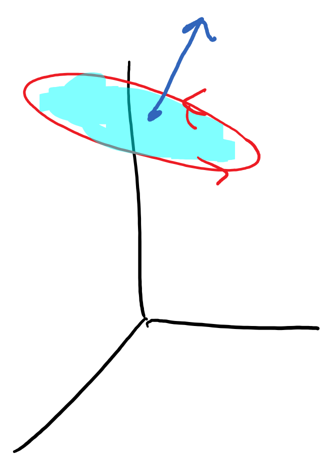
<u>surface</u>

$x = x$

$y = y$

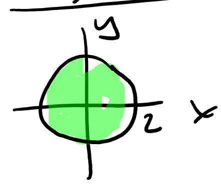
$z = 3 - y$

$$\mathbf{r}_x \times \mathbf{r}_z = \langle 0, 1, 1 \rangle$$



$$\iint_S \text{curl}\,\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y)\, dA$$

<u>Region D</u>



$x = r\cos\theta$

$y = r\sin\theta$

$$\iint_D 1 + 2y \, dA$$

$$\int^{2\pi} \int^{2}$$

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{2} (1+2\cdot r\sin\theta)\, r \; dr\, d\theta$$

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\left(r+2r^2\sin\theta\right) dr\, d\theta = \int_{\theta=0}^{2\pi}\left(\frac{r^2}{2}+\frac{2r^3}{3}\sin\theta\right)\Big|_0^2 d\theta$$

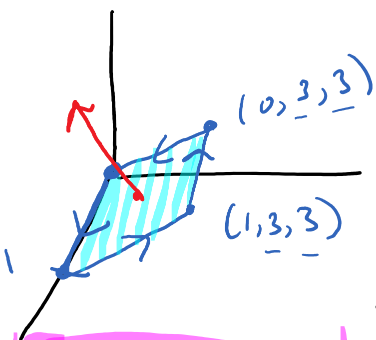
$$= \int_0^{2\pi}\left(2+\frac{16}{3}\sin\theta\right) d\theta = \left(2\theta - \frac{16}{3}\cos\theta\right)\Big|_0^{2\pi}$$

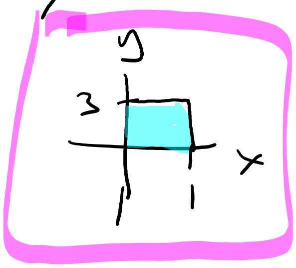
$$= 4\pi - \frac{16}{3} - \left(0 - \frac{16}{3}(1)\right) = 4\pi - \frac{16}{3} + \frac{16}{3}$$

$$= 4\pi$$

Example: Find the work performed by the force field $\mathbf{F} = \langle 3x^8, 4xy^3, y^2x\rangle$ on a particle that traverses the curve C in the plane $z = y$ consisting of 4 line segments starting at the point $(0,0,0)$ to $(1,0,0)$ to $(1,3,3)$ to $(0,3,3)$ to $(0,0,0)$.

$$\text{work} = \int_C F \cdot dr = \iint_S \text{curl}\, F \cdot dS$$





surface

$X = x$

$y = y$

$z = y = f(x,y)$

$$r_x \times r_y = \langle 0, -1, 1\rangle$$

orientation works ✓

$$F = \langle \underset{P}{3x^8}, \underset{Q}{4xy^3}, \underset{R}{y^2x}\rangle$$

$\text{curl}\, \mathbf{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\rangle$

$$\text{curl}\, F = \langle 2yx - 0, 0 - y^2, 4y^3 - 0\rangle$$
$$= \langle 2yx, -y^2, 4y^3\rangle$$

$$\text{curl}\, F \cdot (r_x \times r_y) = 0(2yx) + (-1)(-y^2) + 1(4y^3)$$
$$= y^2 + 4y^3$$

$$\int_C F \cdot dr = \iint_S \text{curl}\, F \cdot dS = \iint_D y^2 + 4y^3\, dA$$

C $\bar{S}$ D

$$= \int_{x=0}^{1}\int_{y=0}^{3} y^2 + 4y^3 \, dy \, dx$$

$$= \int_{x=0}^{1} \left(\frac{y^3}{3} + y^4 \right)\Big|_0^3 \, dx$$

$$= \int_{x=0}^{1} (9 + 81) \, dx = \int_0^1 90 \, dx = 90x \Big|_0^1$$

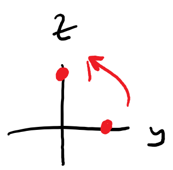
$$= 90$$

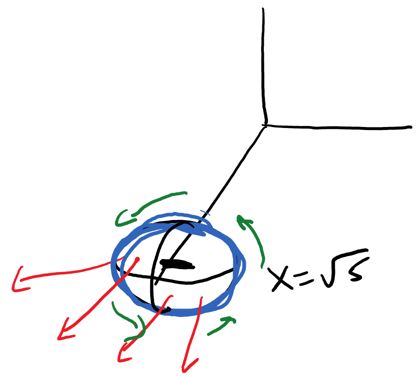
curl $\mathbf{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$

Example: Use Stokes' Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ if S is the part of the hemisphere $x = \sqrt{9 - y^2 - z^2}$ that lies inside the cylinder $y^2 + z^2 = 4$, oriented in the direction of the positive x–axis.

$\mathbf{F} = \langle xy,\ xy^2 z,\ z + x \rangle$

$$\iint_S \text{curl} F \cdot dS = \int_C F \cdot dr$$





$x^2+y^2+z=9 \qquad y^2+z^2=4$

$x^2+4=9$

$x^2=5$

$x=\sqrt{5}$

path C

$x=\sqrt{5}$

$y=2\cos\theta$

$z=2\sin\theta$

$r(\theta) = \langle \sqrt{5},\ 2\cos\theta,\ 2\sin\theta \rangle$

This is correct. direction

$r'(\theta) = \langle 0,\ -2\sin\theta,\ 2\cos\theta \rangle$

$F = \langle xy,\ xy^2 z,\ z+x \rangle$

$F = \langle \sqrt{5}\cdot 2\cos\theta,\ \sqrt{5}\cdot 8\cos^2\theta \sin\theta,\ 2\sin\theta + \sqrt{5} \rangle$

$$\underbrace{\iint_S \text{curl} F \cdot dS = \int_C F \cdot dr}_{\text{Stokes thrm}}$$

$$= \int_0^{2\pi} -16\sqrt{5}\cos^2\theta \sin^2\theta + 4\sin\theta\cos\theta + 2\sqrt{5}\cos\theta \, d\theta$$

$\quad 0$

$$= -4\sqrt{5}\pi + 0 + 0$$

$$= -4\sqrt{5}\pi$$

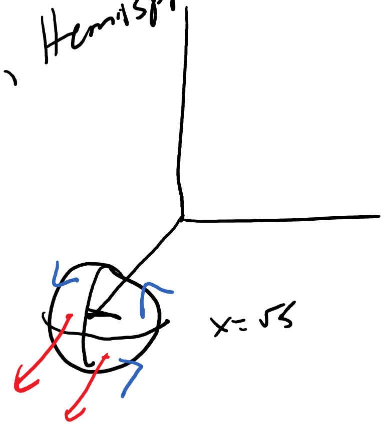
S_1 Hemisphere

path C

$x = \sqrt{5}$

$y = 2\cos\theta$

$z = 2\sin\theta$



$x = \sqrt{5}$ $\quad y^2 + z^2 = 4$

$$\iint_{S_1} \text{curl} F \cdot dS_1 = \int_C F \cdot dr$$

S_2 disk of Radius 2 at $x = \sqrt{5}$ centered on x-axis

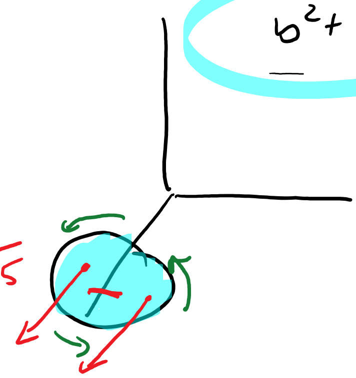
$y^2 + z^2 = 4$

path C

$x = \sqrt{5}$

$y = 2\cos\theta$

$z = 2\sin\theta$



$x = \sqrt{5}$

$$\iint_{S_2} \text{curl} F \cdot dS_2 = \int_C F \cdot dr$$

$$F = \langle xy,\ xzy^2,\ z + x \rangle$$

$$\iint_{S_1} \text{curl} F \cdot dS_1 = \int_C F \cdot dr = \iint_{S_2} \text{curl} F \cdot dS_2$$

$$\text{curl} F = \langle -xy^2,\ x - 1,\ y^2 z \rangle$$

Surface S_2 disk $\quad x = \sqrt{5} \quad y = y \quad z = z$

$$r(y, z) = \langle \sqrt{5},\ y,\ z \rangle$$

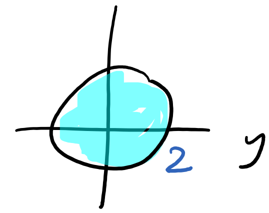
$$\text{cross product} = \langle 1,\ 0,\ 0 \rangle$$

orientation is correct

$$\text{curl} F \cdot \text{cross product} = -xy^2 = -\sqrt{5}\, y^2$$

z

$$\iint_{S_2} \text{curl}\, F \cdot dS = \iint_D -\sqrt{5}\, y^2 \, dA$$



$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} -\sqrt{5}\, r^2 \cos^2\theta \cdot r \, dr \, d\theta$$

$$= \underline{\int_0^{2\pi} \cos^2\theta \, d\theta} \cdot \int_{r=0}^{2} -\sqrt{5}\, r^3 \, dr$$

$$= -4\sqrt{5}\,\pi$$