Section 16.7: Surface Integrals

Definition: If S is parametrized by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, then the **surface integral of f over the surface S** is

$$\iint_S \underline{f(x, y, z)}dS = \iint_D f(\mathbf{r}(u, v))|\mathbf{r_u} \times \mathbf{r_v}|dA$$

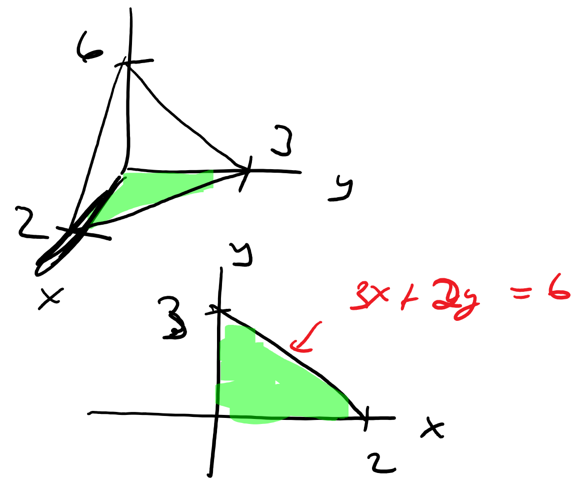
where D is a region in the $uv-$plane.

Application: If the function is the density at the points of the surface then the surface integral over S computes the mass of the surface.

$$\text{mass: } m = \iint_S \rho(x, y, z)dS$$

$$dS = |r_x \times r_y|\, dA$$

Example: Evaluate $\displaystyle\iint_S xz\,dS$ where S is the part of the plane $\underbrace{3x+2y+z=6}$ in the first octant.



$$r(x,y) = \langle x,\ y,\ 6-3x-2y \rangle$$

$$\langle -f_x,\ -f_y,\ 1\rangle$$

$$r_x \times r_y = \langle 3,\ 2,\ 1\rangle$$

$$|r_x \times r_y| = \sqrt{9+4+1} = \sqrt{14}$$

$$0 \le x \le 2$$

$$0 \le y \le \frac{6-3x}{2}$$

$$\iint_S xz\,dS =$$

$$= \int_{x=0}^{2}\int_{y=0}^{\frac{6-3x}{2}} x(6-3x-2y)\sqrt{14}\,dy\,dx$$

$$= \sqrt{14}\int_{x=0}^{2}\int_{y=0}^{\frac{6-3x}{2}} 6x-3x^2-2xy\,dy\,dx$$

$$= \sqrt{14}\int_{x=0}^{2} \left(6xy-3x^2y-xy^2\right)\Big|_0^{\frac{6-3x}{2}}\,dx$$

$$= \int^{2}$$

$$= \sqrt{14} \int_0^2 6x \frac{(6-3x)}{2} - 3x^2 \frac{(6-3x)}{2} - x\left(\frac{6-3x}{2}\right)^2 \, dx$$

$$= \sqrt{14} \int_0^2 18x - 9x^2 - \frac{18x^2}{2} + \frac{9x^3}{2} - \frac{x}{4}\left(36 - 36x + 9x^2\right) dx$$

$$= \sqrt{14} \int_0^2 18x - 18x^2 + \frac{9}{2}x^3 - 9x + 9x^2 - \frac{9}{4}x^3 \, dx$$

$$= \sqrt{14} \int_0^2 9x - 9x^2 + \frac{9}{4}x^3 \, dx$$

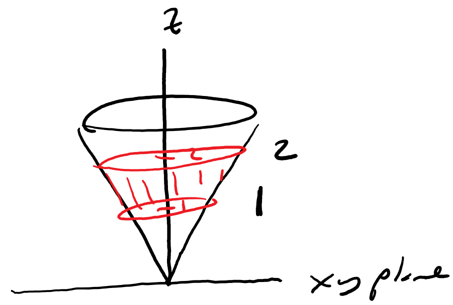
$$= \sqrt{14} \left[\frac{9x^2}{2} - 3x^3 + \frac{9}{16}x^4 \right]_0^2$$

$$= \sqrt{14} \left[18 - 24 + 9 \right] = \sqrt{14}\,(3) = \underline{\underline{3\sqrt{14}}}$$

Example: Compute $\iint_S y^2 z^2 \, dS$ where S is the part of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$.

$2 = \sqrt{x^2+y^2}$

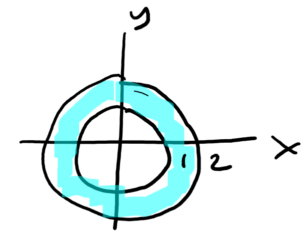
$2^2 = x^2+y^2$



$x = r\cos\theta$
$y = r\sin\theta$
$z = \sqrt{r^2} = r$

$1 \le r \le 2$

$0 \le \theta \le 2\pi$



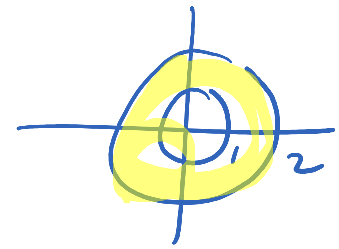
$r(r, \theta) = \langle r\cos\theta,\ r\sin\theta,\ r \rangle$

$$r_r \times r_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -r\cos\theta, -r\sin\theta, r \rangle$$

$$|r_r \times r_\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{r^2 + r^2} = \sqrt{2r^2} = r\sqrt{2}$$

$$\iint_S y^2 z^2 \, dS = \int_{\theta=0}^{2\pi} \int_{r=1}^{2} r^2 \sin^2(\theta)\, r^2\, \sqrt{2}\ r \, dr\, d\theta = \ldots \frac{21\pi\sqrt{2}}{2}$$

Example: Compute $\iint\limits_S y^2 z^2 dS$ where S is the part of the cone $z = \sqrt{x^2+y^2}$ between the planes $z = 1$ and $z = 2$.



Using $x = x \qquad y = y \qquad z = \sqrt{x^2+y^2}$

$r(x,y) = \left\langle x,\ y,\ \sqrt{x^2+y^2} \right\rangle$

$-f_x \qquad -f_y \qquad 1$

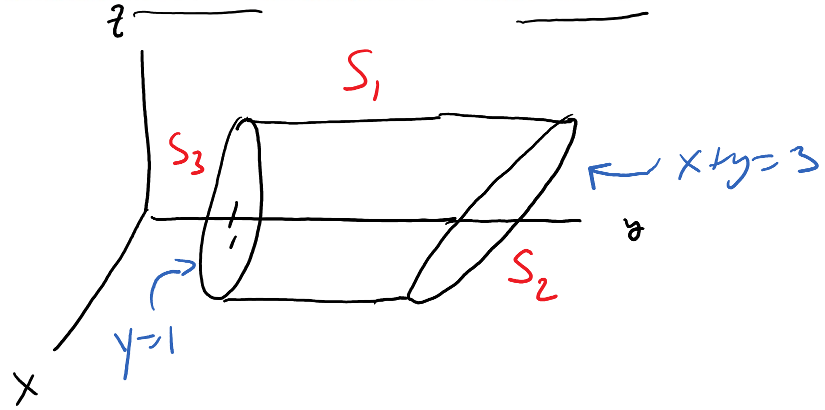
$$r_x \times r_y = \left\langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1 \right\rangle$$

$$|r_x \times r_y| = \sqrt{\boxed{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}} + 1} = \sqrt{2}$$

$$\iint\limits_S y^2 z^2 dS = \iint\limits_D y^2 (\sqrt{x^2+y^2})^2 \sqrt{2} \ \ dA =$$

$$\int_{\theta=0}^{2\pi} \int_{r=1}^{2} r^2 \sin^2(\theta)\, r^2 \ \sqrt{2} \ \ r \ dr d\theta = \ldots \frac{21\pi\sqrt{2}}{2}$$

Example: Compute $\iint_S xy\,dS$ where S is the boundary of the region enclosed by the cylinder $x^2+z^2=1$ and the planes $y=1$ and $x+y=3$



S_1)

$x = 1\cos\theta$
$y = y$
$z = 1\sin\theta$

$0 \le \theta \le 2\pi$
$1 \le y \le 3 - x = 3 - \cos\theta$

$r(y,\theta) = \langle \cos\theta, y, \sin\theta \rangle$

$r_y = \langle 0, 1, 0 \rangle$

$r_\theta = \langle -\sin\theta, 0, \cos\theta \rangle$

$r_\theta \times r_y = \langle -\cos\theta, 0, -\sin\theta \rangle$

$$|r_\theta \times r_y| = \sqrt{(-\cos\theta)^2 + 0 + (-\sin\theta)^2} = \sqrt{1} = 1$$

$$\iint_{S_1} xy\,dS_1 = \iint_D \cos\theta \cdot y \cdot 1\,dA$$

$$= \int_{\theta=0}^{2\pi} \int_{y=1}^{3-\cos\theta} y\cos\theta\,dy\,d\theta$$

$\theta=0 \quad y=1$

$$= \int_{\theta=0}^{2\pi} \frac{y^2}{2} \cos\theta \Big|_1^{3-\cos\theta} \, d\theta$$

$$= \int_0^{2\pi} \frac{(3-\cos\theta)^2 \cos\theta}{2} - \frac{1}{2}\cos\theta \, d\theta$$

$$= \int_0^{2\pi} \frac{(9-6\cos\theta+\cos^2\theta)\cos\theta}{2} - \frac{1}{2}\cos\theta \, d\theta$$

$$= \int_0^{2\pi} \frac{9\cos\theta}{2} - \frac{6\cos^2\theta}{2} + \frac{1}{2}\cos^3\theta - \frac{1}{2}\cos\theta \, d\theta$$

$$= \int_0^{2\pi} 4\cos\theta - 3\cdot\frac{1}{2}(1+\cos 2\theta) + \frac{1}{2}\cos^2\theta\cos\theta \, d\theta$$

$$= \int_0^{2\pi} 4\cos\theta - \frac{3}{2} - \frac{3}{2}\cos 2\theta + \frac{1}{2}(1-\sin^2\theta)\cos\theta \, d\theta$$

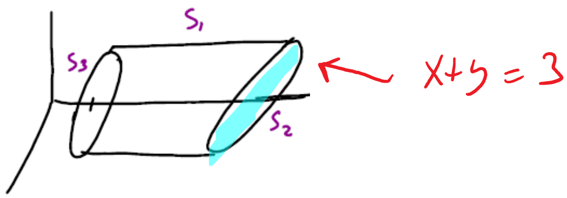
$$= \int_0^{2\pi} 4\cos\theta - \frac{3}{2} - \frac{3}{2}\cos 2\theta + \frac{1}{2}\cos\theta - \frac{1}{2}\sin^2\theta\cos\theta \, d\theta$$

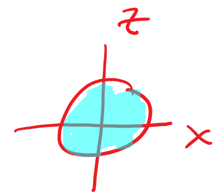
$/_0$

$$= \left(4.5 \sin\theta - \frac{3}{2}\theta - \frac{3}{2}\cdot\frac{1}{2}\sin(2\theta) - \frac{1}{2}\cdot\frac{1}{3}\sin^3\theta \right)\Big|_0^{2\pi}$$

$$= -\frac{3}{2}\cdot 2\pi - (0) = -3\pi$$

Example: Compute $\iint_S xy\,dS$ where S is the boundary of the region enclosed by the cylinder $x^2+z^2=1$ and the planes $y=1$ and $x+y=3$





$\underline{S_2}$

$x = x$

$y = 3-x$

$z = z$

$$-f_x,\ 1,\ -f_z$$

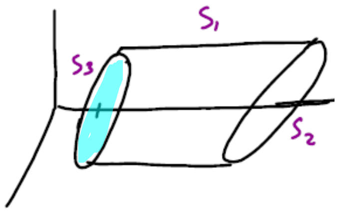
$$r_x \times r_z = \langle 1, 1, 0 \rangle$$

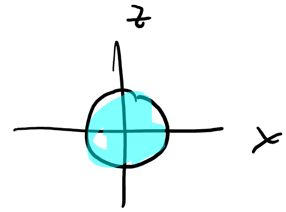
$$|r_x \times r_z| = \sqrt{2}$$

$$\iint_{S_2} xy\,ds = \iint_D x(3-x)\sqrt{2}\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\cos\theta(3-r\cos\theta)\sqrt{2}\,r\,dr\,d\theta$$

$$= -\frac{\sqrt{2}}{4}\pi$$

Example: Compute $\iint\limits_S xy\,dS$ where S is the boundary of the region enclosed by the cylinder $x^2+z^2=1$ and the planes $y=1$ and $x+y=3$





S_3

$x = x$ $\quad r_x \times r_z = \langle 0, 1, 0 \rangle$

$y = 1 \leftarrow$

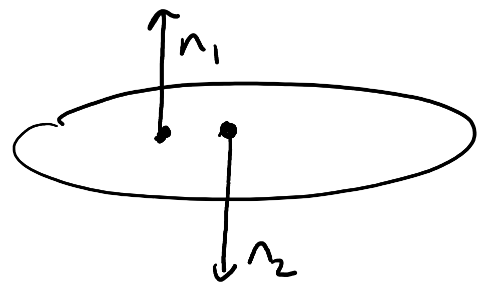
$z = z$ $\quad |r_x \times r_z| = \sqrt{1} = 1$

$$\iint\limits_{S_3} xy\,dS_3 = \iint\limits_D x(1)\cdot 1\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\cos\theta \cdot r\,dr\,d\theta$$

$$= \underbrace{\int_{\theta=0}^{2\pi}\cos\theta\,d\theta}\cdot\int_{r=0}^{1} r^2\,dr = 0$$

Answer: $-3\pi + -\frac{\sqrt{2}}{4}\pi + 0$

Pg 5: surface integrals over vector fields



Let S be a surface parametrized by $\mathbf{r}(u, v)$. If S has a tangent plane at every point on S (except at any boundary points), then there are two unit normal vectors at every point.

$$\mathbf{n}_1 = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \text{ and } \mathbf{n}_2 = \frac{\mathbf{r}_v \times \mathbf{r}_u}{|\mathbf{r}_v \times \mathbf{r}_u|}$$

The normal vector provides an orientation for S and S is called an **oriented surface**

For a surface defined by $z = g(x, y)$, then $\mathbf{n} = \dfrac{\langle -g_x, -g_y, 1\rangle}{\sqrt{1 + (g_x)^2 + (g_y)^2}}$

Since the $\mathbf{k}$ component is positive, this gives the upward orientation of the surface.

Note: For a closed surface, a surface that is the boundary of a solid region(volume), **positive orientation** is where the normal vectors point outward from the region.

Definition: If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, the the **surface integral of F over S** is

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

This integral is also called the **flux** of $\mathbf{F}$ across S.

Note: If S is parametrized by $\mathbf{r}(u,v)$, then $\mathbf{n} = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$

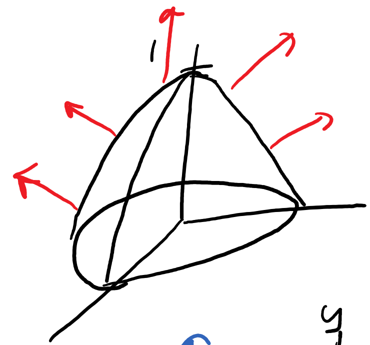
This gives $d\vec{\mathbf{S}} = \mathbf{n}\, dS = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, dS = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, |\mathbf{r}_u \times \mathbf{r}_v| dA = (\mathbf{r}_u \times \mathbf{r}_v) dA$

Thus

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$$

Note: choose the cross product that gives the correct orientation for the problem.

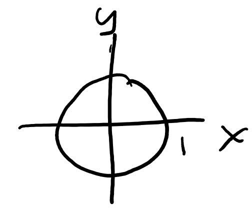
Example: Let S be the part of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane with upward orientation. Find the flux of $\mathbf{F} = \langle x, y, 3z \rangle$ across S.



$\underline{S|}$ $r(x,y) = \langle x, y, 1-x^2-y^2 \rangle$

$r_x \times r_y = \langle 2x, 2y, 1 \rangle$

upward orientation



$0 \le \theta \le 2\pi$

$0 \le r \le 1$

$F = \langle x, y, 3-3x^2-3y^2 \rangle$

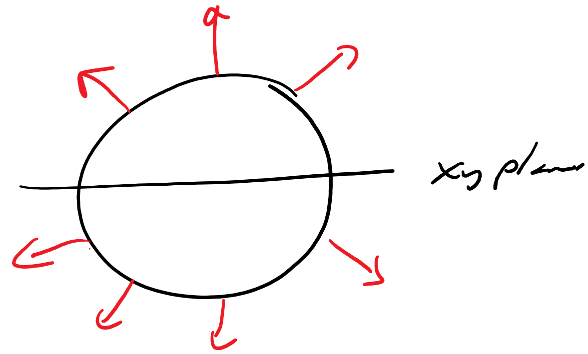
$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D F \cdot (r_x \times r_y)\, dA$$

$$= \iint_D 2x^2 + 2y^2 + 3 - 3x^2 - 3y^2 \; dA$$

$$= \iint_D 3 - x^2 - y^2 \; dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (3-r^2)\, r \, dr \, d\theta$$

$$= \frac{5\pi}{2}$$

Example: Let S be the sphere $x^2+y^2+z^2=16$ with a positive orientation and $\mathbf{F}=\langle 0,0,z\rangle$. Evaluate $\iint\limits_S \mathbf{F}\cdot d\mathbf{S}$



$x = 4\sin\phi\cos\theta$

$y = 4\sin\phi\sin\theta$

$z = 4\cos\phi$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

$$r_\phi \times r_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4\cos\phi\cos\theta & 4\cos\phi\sin\theta & -4\sin\phi \\ -4\sin\phi\sin\theta & 4\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$r_\phi \times r_\theta = \left\langle 16\sin^2\phi\cos\theta,\ -16\sin^2\phi\sin\theta,\ 16\sin\phi\cos\phi\cos^2\theta + 16\sin\phi\cos\phi\sin^2\theta \right\rangle$$

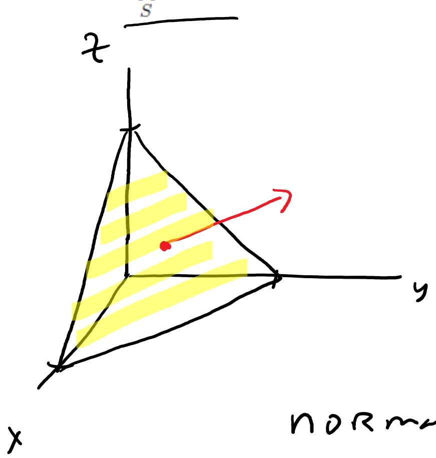
$$r_\phi \times r_\theta = \left\langle 16\sin^2\phi\cos\theta,\ -16\sin^2\phi\sin\theta,\ \underline{16\sin\phi\cos\phi} \right\rangle \quad \checkmark$$

$$F\cdot(r_\phi \times r_\theta) = 16z\sin\phi\cos\phi = 16\cdot 4\cos^2\phi\sin\phi$$
$$= 4^3\cos^2\phi\sin\phi$$

$$\iint\limits_S F\cdot d\vec{S} = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} 4^3\cos^2\phi\sin\phi \, d\phi\, d\theta$$

$$= \frac{4^4\pi}{3}$$

Example: Let S be the closed surface of a Tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, i.e the surface of the solid in the first octant that is formed by the plane $x+y+z=1$ and the three coordinate planes. Let $\mathbf{F} = \langle y, z-y, x\rangle$. and use positive orientation.
Evaluate $\iint\limits_S \mathbf{F}\cdot d\mathbf{S}$



Sides

S_1	slant
S_2	xy plane
S_3	xz plane
S_4	yz plane.

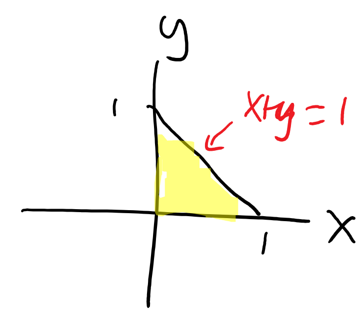
normal vectors to point out.

S_1 $\qquad r(x,y) = \langle x, y, 1-x-y\rangle$

$r_x \times r_y = \langle 1, 1, 1\rangle$ $\qquad$ orientation ✓

$$F = \langle y, z-y, x\rangle = \langle y, 1-x-2y, x\rangle$$

$$\iint_{S_1} F\cdot dS_1 = \iint_D y+1-x-2y+x \, dA$$



$$= \iint_D 1-y \, dA$$

$0 \le y \le 1$
$0 \le x \le 1-y$

$$= \int_{y=0}^{1}\int_{x=0}^{1-y} 1-y \, dx\, dy$$

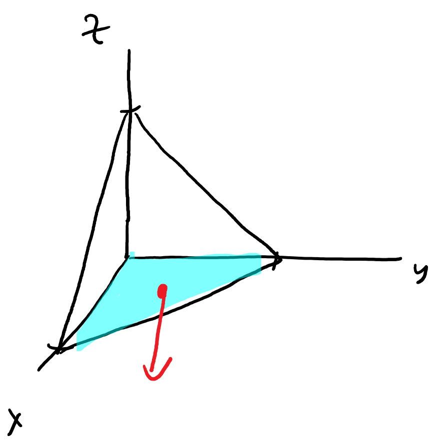
$y=0 \quad x=0$

$$= \int_{y=0}^{1} (1-y)\, x \Big|_{0}^{1-y} dy = \int_{y=0}^{1} (1-y)(1-y)\, dy$$

$$= \int_{y=0}^{1} 1 - 2y + y^2 \, dy = y - y^2 + \frac{y^3}{3} \Big|_{0}^{1}$$

$$= 1 - 1 + \frac{1}{3} = \frac{1}{3}$$

Example: Let S be the closed surface of a Tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, i.e the surface of the solid in the first octant that is formed by the plane $x+y+z=1$ and the three coordinate planes. Let $\mathbf{F} = \langle y, z-y, x\rangle$. and use positive orientation.
Evaluate $\iint\limits_S \mathbf{F}\cdot d\mathbf{S}$



Sides

S_1	slant
S_2	xy plane
S_3	xz plane
S_4	yz plane.

S_2: $x = x$, $y = y$, $z = 0 = f(x,y)$

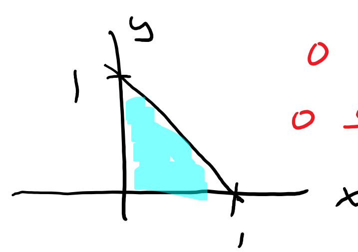
$r_x \times r_y = \langle 0, 0, 1\rangle$

(not the correct orientation

use $\langle 0, 0, -1\rangle$ as the normal vector.

$F = \langle y, z-y, x\rangle$

$F = \langle y, -y, x\rangle$



$0 \le x \le 1$

$0 \le y \le 1-x$

$$\iint\limits_{S_2} F\cdot d\vec{S} = \iint\limits_D -x\, dA = \int_{x=0}^{1}\int_{y=0}^{1-x} -x\, dy\, dx$$

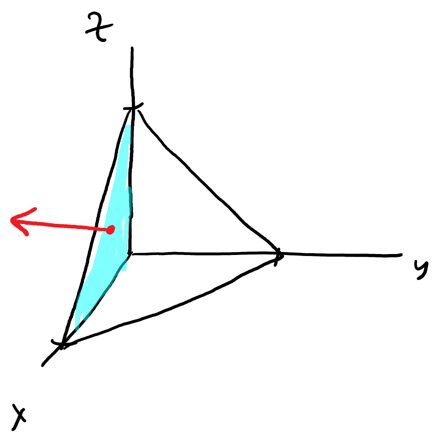
$$= \int_{x=0}^{1} -xy\Big|_0^{1-x} dx = \int_{x=0}^{1} -x(1-x)\, dx$$

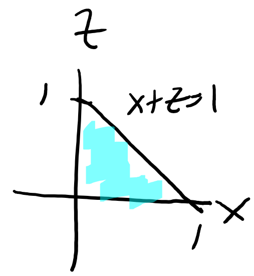
$x=0$ $\qquad\qquad$ $x=0$

$$= \int_{x=0}^{1} -x + x^2 \, dx = -\frac{x^2}{2} + \frac{x^3}{3} \Big|_0^1$$

$$= -\frac{1}{2} + \frac{1}{3} = -\frac{3}{6} + \frac{2}{6} = -\frac{1}{6}$$

Example: Let S be the closed surface of a Tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, i.e the surface of the solid in the first octant that is formed by the plane $x+y+z=1$ and the three coordinate planes. Let $\mathbf{F} = \langle y, z-y, x\rangle$. and use positive orientation.
Evaluate $\iint\limits_S \mathbf{F}\cdot d\mathbf{S}$





$0 \le z \le 1$
$0 \le x \le 1-z$

Sides	
S_1	slant
S_2	xy plane
S_3	xz plane
S_4	yz plane.

S_3

$x = x$
$y = 0 = f(x,z)$
$z = z$

$r(x,z) = \langle x, 0, z\rangle$

$r_x \times r_z = \langle 0, 1, 0\rangle$

use $\underline{\langle 0, -1, 0\rangle}$

$F = \langle y, z-y, x\rangle$

$F = \langle 0, z, x\rangle$

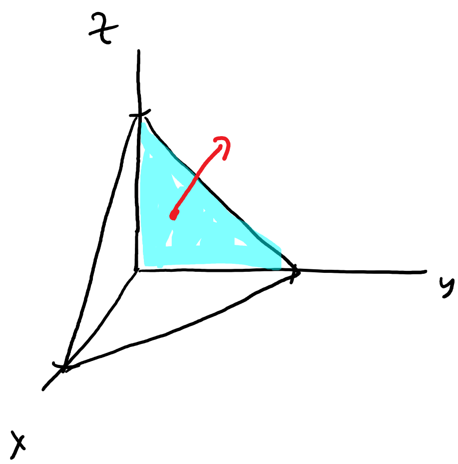
$$\iint\limits_{S_3} F\cdot d\vec{S}_3 = \iint\limits_D -z \, dA = \int_{z=0}^{1}\int_{x=0}^{1-z} -z \, dx \, dz$$

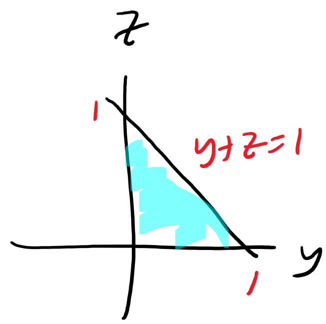
$$= \int_{z=0}^{1} -zx \Big|_0^{1-z} dz = \int_{z=0}^{1} -z(1-z) \, dz$$

$$= \int^{1} -z + z^2 \, dz = \ldots = -\frac{1}{6}$$

$$= \int_0^1 -z + z^2 \quad dz = \dots = -\frac{1}{6}$$

Example: Let S be the closed surface of a Tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, i.e the surface of the solid in the first octant that is formed by the plane $x+y+z=1$ and the three coordinate planes. Let $\mathbf{F} = \langle y, z-y, x\rangle$. and use positive orientation.
Evaluate $\iint\limits_S \mathbf{F}\cdot d\mathbf{S}$





$0 \le y \le 1$

$0 \le z \le 1-y$

Sides	
S_1	slant
S_2	xy plane
S_3	xz plane
S_4	yz plane.

S_4) $x = 0 = f(y,z)$

$y = y$

$z = z$

$r_y \times r_z = \langle 1, 0, 0\rangle$

wrong orientation

use $\langle -1, 0, 0\rangle$

$F = \langle y, z-y, x\rangle$

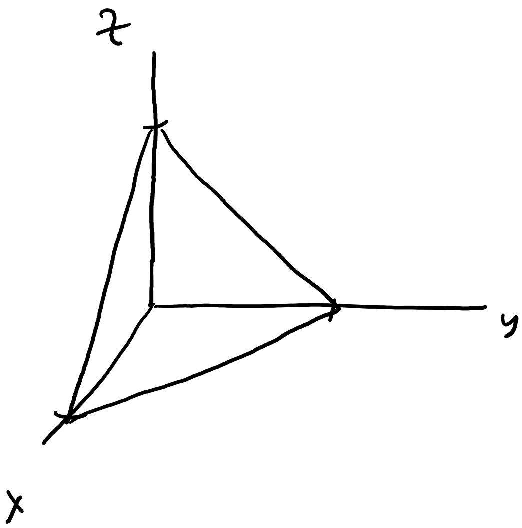
$F = \langle y, z-y, 0\rangle$

$$\iint\limits_{S_4} F\cdot dS_4 = \iint\limits_D -y\, dA = \int_{y=0}^{1}\int_{z=0}^{1-y} -y\, dz\, dy$$

$$= \int_0^1 -yz\Big|_0^{1-y} dy = \int_0^1 -y(1-y)\, dy$$

$$= \int_0^1 -y + y^2 \, dy = \cdots = -\frac{1}{6}$$

Example: Let S be the closed surface of a Tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, i.e the surface of the solid in the first octant that is formed by the plane $x+y+z=1$ and the three coordinate planes. Let $\mathbf{F} = \langle y, z-y, x\rangle$. and use positive orientation.
Evaluate $\iint\limits_S \mathbf{F}\cdot d\mathbf{S}$



Sides	
S_1	slant
S_2	xy plane
S_3	xz plane
S_4	yz plane.

$$\iint\limits_S \mathbf{F}\cdot d\mathbf{S} = \underset{S_1}{\frac{1}{3}} - \underset{S_2}{\frac{1}{6}} - \underset{S_3}{\frac{1}{6}} - \underset{S_4}{\frac{1}{6}} = -\frac{1}{6}$$