

Section 16.5: Curl and Divergence

Definition: The del operator, denoted ∇ , is defined as $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

Definition: If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field on \mathbb{R}^3 and the partial derivatives of P , Q , and R all exist, then the curl of \mathbf{F} is the vector field on \mathbb{R}^3 defined by

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\text{curl } \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$\mathbf{F} = \langle P, Q, R \rangle$$

Note: The curl \mathbf{F} relates to rotation of a fluid at point P around the axis that points in the same direction of curl \mathbf{F} .

Note: If the curl $\mathbf{F} = \mathbf{0}$ at point P , then \mathbf{F} is called irrotational at P .

P Q R

Example: Find the curl of the vector field $\mathbf{F} = \langle x^2y, yz^2, zx^2 \rangle$

$$\text{curl } \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$\begin{aligned} \text{curl } \mathbf{F} &= \langle 0 - 2yz, 0 - 2zx, 0 - x^2 \rangle \\ &= \langle -2yz, -2zx, -x^2 \rangle \end{aligned}$$

Theorem: If f is a function of three variables that has continuous second-order partial derivatives, then $\text{curl}(\nabla f) = \mathbf{0}$.

$$\nabla f = \langle P, Q, R \rangle = \langle f_x, f_y, f_z \rangle$$

$$\text{curl}(\nabla f) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$\text{curl}(\nabla f) = \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle = \langle 0, 0, 0 \rangle$$

$$R = f_z$$

$$R_y = f_{zy}$$

Theorem: If \mathbf{F} is a vector field defined on all of \mathbb{R}^3 whose component functions have continuous partial derivatives and $\text{curl } \mathbf{F} = \mathbf{0}$, then \mathbf{F} is a conservative vector field.

P & n

Example: Determine if the vector field is conservative. $\mathbf{F} = \langle zx, xy, yz \rangle$.

$$\text{curl } \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

not conservative

$$\text{Curl } \mathbf{F} = \langle z - 0, x - 0, y - 0 \rangle$$

$$= \langle z, x, y \rangle$$

Pg 4: divergence

Definition: If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field on \mathbb{R}^3 and P_x , Q_y and R_z exist, then the divergence of \mathbf{F} is the scalar function given by

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \underline{\underline{\nabla \cdot \mathbf{F}}} = P_x + Q_y + R_z$$

Note: If \mathbf{F} is a velocity field for a fluid, then $\operatorname{div} \mathbf{F}$ at a point measures the tendency of the fluid to diverge from that point. $\operatorname{div} \mathbf{F}$ positive(negative) means move away(towards).

$P \quad Q \quad R$

Example: Compute the $\operatorname{div} \mathbf{F}$ where $\mathbf{F} = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$.

$$\begin{aligned} \operatorname{div} \mathbf{F} &= P_x + Q_y + R_z \\ &= 0 + 2xz^3 + 6xy^2z \end{aligned}$$

Theorem: If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field on \mathbb{R}^3 and P , Q , and R have continuous second-order partial derivatives, then

$$\operatorname{div}(\operatorname{curl} \mathbf{F}) = \operatorname{div} \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$\operatorname{div} \operatorname{curl} \mathbf{F} = \underbrace{R_{yx}} - \underbrace{Q_{zx}} + \underbrace{P_{zy}} - \underbrace{R_{xy}} + \underbrace{Q_{xz}} - \underbrace{P_{yz}}$$

$$\operatorname{div} \operatorname{curl} \mathbf{F} = 0$$

Example: Is there a vector field \mathbf{G} on \mathbb{R}^3 such that $\operatorname{curl} \mathbf{G} = \langle \underline{yz}, \underline{xyz}, \underline{zy} \rangle$?

$$\operatorname{div} \langle yz, xyz, zy \rangle = 0 + xz + y \neq 0$$

no. There is no vector field \mathbf{G} that gives

$$\operatorname{curl} \mathbf{G} = \langle yz, xyz, zy \rangle$$

Pg 6: interesting information

Example: Compute $\text{curl } \mathbf{F}$ if \mathbf{F} is a vector field on \mathbb{R}^2 .

Suppose that $F = \langle P, Q \rangle$. Lets expand this to three dimensions by letting the z component, R , be zero.

i.e. $F = \langle P, Q, 0 \rangle$.

$$\text{Thus } \text{curl} \mathbf{F} = \langle \underbrace{R_y - Q_z}, \underbrace{P_z - R_x}, \underbrace{Q_x - P_y} \rangle = \langle 0, 0, \underbrace{Q_x - P_y} \rangle$$

$$\underline{k = \langle 0, 0, 1 \rangle}$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy = \iint_D Q_x - P_y dA = \iint_D \text{curl } \mathbf{F} \cdot \mathbf{k} dA$$