Section 16.4: Green's Theorem

The **positive orientation** of a simple closed curve C refers to a single <u>counterclockwise</u> traversal of C.

Green's Theorem: Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

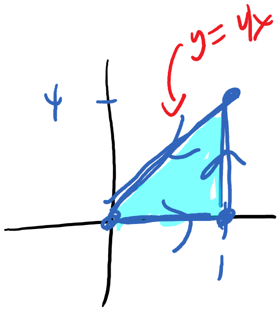
$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D Q_x - P_y \, dA$$

Alternate notations: When $F = \langle P, Q\rangle$ and curve given by $r(t)$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy = \oint_C P\,dx + Q\,dy$$

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Example: Evaluate $\oint_C \overset{P}{x^2y}\,dx + \overset{Q}{x}\,dy$ where C is the triangular path from $(0,0)$ to $(1,0)$ to $(1,4)$ to (0.0).



Counter clockwise ✓

Simple closed path ✓

Green's Thrm can be used.

$P = x^2 y$ $\qquad P_y = x^2$

$Q = x$ $\qquad Q_x = 1$

$$\int_C P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA$$

$$= \iint_D 1 - x^2 \, dA$$

$0 \leq x \leq 1$

$0 \leq y \leq 4x$

$$= \int_{x=0}^{1} \int_{y=0}^{4x} 1 - x^2 \, dy\,dx = \int_{x=0}^{1} y - yx^2 \Big|_0^{4x} dx$$

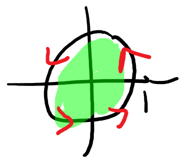
$$= \int_{x=0}^{1} 4x - 4x\,x^2 \, dx = \int_{x=0}^{1} 4x - 4x^3 \, dx$$

$$= 2x^2 - x^4 \Big|_0^1 = 2 - 1 = \boxed{1}$$

Example: Suppose a particle travels one revolution counter-clockwise around the unit circle under the force field $\mathbf{F}(x, y) = \langle e^x - y^3, \cos(y) + x^3 \rangle$. Find the work done by the field.

$P = e^x - y^3 \qquad P_y = -3y^2$

$Q = \cos y + x^3 \qquad Q_x = 3x^2$



positive orientation ✓

closed path ✓

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

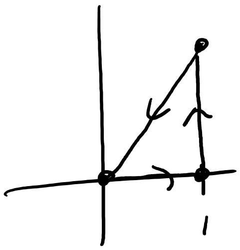
$$W = \int_C F \cdot dr$$

$$= \iint_D 3x^2 - -3y^2 \, dA = \iint_D 3x^2 + 3y^2 \, dA$$

$$= \int_0^{2\pi} \int_0^1 3r^2 \cdot r \, dr \, d\theta = \int_0^{2\pi} 1 \, d\theta \cdot \int_0^1 3r^3 \, dr$$

$$= 2\pi \cdot \frac{3r^4}{4} \Big|_0^1 = 2\pi \cdot \frac{3}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$$

Example: Evaluate $\oint_C \overbrace{\left(x^2y + \frac{1}{2}y^2 + e^{\sin(x)}\right)}^{P} dx + \overbrace{\left(xy + \frac{1}{3}x^3 + x - \arctan(y)\right)}^{Q} dy$
where C is the triangular path from (0, 0) to (1, 0) to (1, 4) to (0.0).



Closed path
positive orientation
green's thrm

$P_y = x^2 + y$

$Q_x = y + x^2 + 1$

$Q_x - P_y = y + x^2 + 1 - (x^2 + y) = 1$

$$\int_C F \cdot dr = \iint_D Q_x - P_y \, dA = \iint_D 1 \, dA = \frac{1}{2}(1)(4) = 2$$

Area Using Line Integrals:

$Q_x - P_y = 1$

Since $\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dA$, we need $\underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1}$

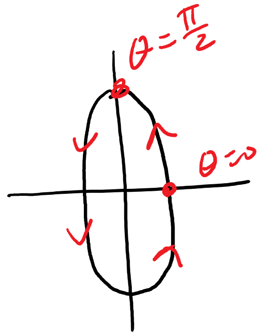
$\iint_D 1\, dA = \int_C F \cdot dr$

$F = \langle P, Q \rangle$

$Q_x - P_y = 1$

Method 1:	Method 2:	Method 3:
$P = 0$ and $Q = x$	$P = -y$ and $Q = 0$	$P = \frac{-1}{2}y$ and $Q = \frac{1}{2}x$

Example: Find the area enclosed by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$



$P = -\frac{1}{2}y$

$Q = \frac{1}{2}x$

$P_y = \frac{-1}{2}$

$Q_x = \frac{1}{2}$

$Q_x - P_y = \frac{1}{2} - \frac{-1}{2} = 1$

$x = 2\cos\theta$

$y = 3\sin\theta$

$0 \le \theta \le 2\pi$

$$\iint_D 1\, dA = \oint_C -\frac{1}{2}y\, dx + \frac{1}{2}x\, dy$$

$$= \int_0^{2\pi} -\frac{1}{2}\, 3\sin\theta\,(-2\sin\theta) + \frac{1}{2}\cdot 2\cos\theta \cdot 3\cos\theta\, d\theta$$

$$= \int_0^{2\pi} 3\sin^2\theta + 3\cos^2\theta\, d\theta = \int_0^{2\pi} 3\, d\theta = 3\theta \Big|_0^{2\pi}$$

$$= 6\pi$$

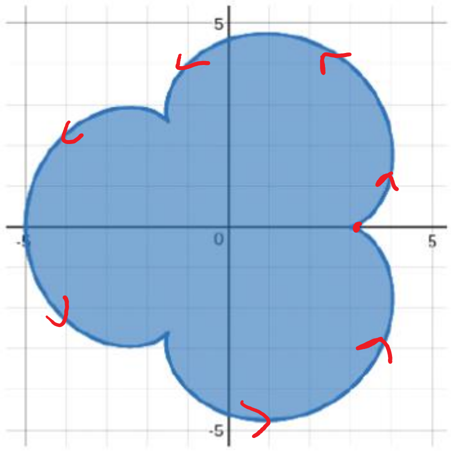
Example: Setup the integral(s) that would give the area of the shaded region shown below. The figure is created with the parametric equations:

$P = 0$

$Q = x$

$x = 4\cos(t) - \cos(4t), \quad y = 4\sin(t) - \sin(4t)$

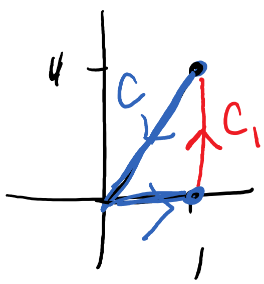
$0 \le t \le 2\pi$



$$\iint_D 1 \, dA = \oint_C 0\,dx + x\,dy = \oint_C x\,dy$$

$$= \int_0^{2\pi} \left[4\cos(t) - \cos(4t)\right] \cdot \left[4\cos(t) - 4\cos(4t)\right] dt$$

Example: Evaluate $\int_C x^2y\,dx + x\,dy$ where C is the path from (1, 4) to (0, 0)to (1, 0)



Closed path ✗
positive orientation ✓

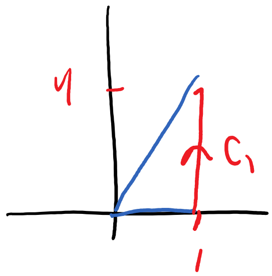
fix the issue by closing the path.

$$\int_{C+C_1} x^2y\,dx + x\,dy$$ can use Green's thm.

by our first example value is 1.

$$\underbrace{\int_{C+C_1} P\,dx + Q\,dy}_{} = \underbrace{\int_{C} P\,dx + Q\,dy}_{} + \underbrace{\int_{C_1} P\,dx + Q\,dy}_{}$$

$$\underbrace{\int_{C} P\,dx + Q\,dy}_{} = \underbrace{\int_{C+C_1} P\,dx + Q\,dy}_{} - \underbrace{\int_{C_1} P\,dx + Q\,dy}_{}$$



$$r(t) = (1-t)\langle 1,0\rangle + t\langle 1,4\rangle$$
$$= \langle 1-t, 0\rangle + \langle t, 4t\rangle$$
$$= \langle 1, 4t\rangle \qquad 0 \le t \le 1$$

$$\int_{C_1} x^2y\,dx + x\,dy = \int_0^1 1\cdot 4\,dt = \int_0^1 4\,dt = 4t\Big|_0^1 = 4$$

$$\int_C P\,dx + Q\,dy = 1 - 4 = -3$$