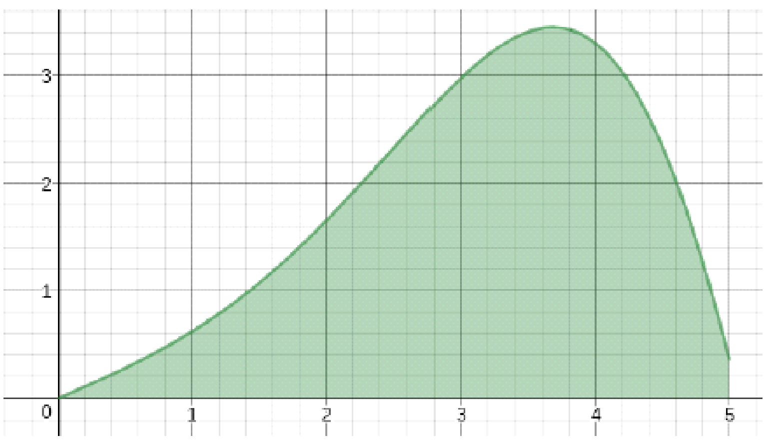
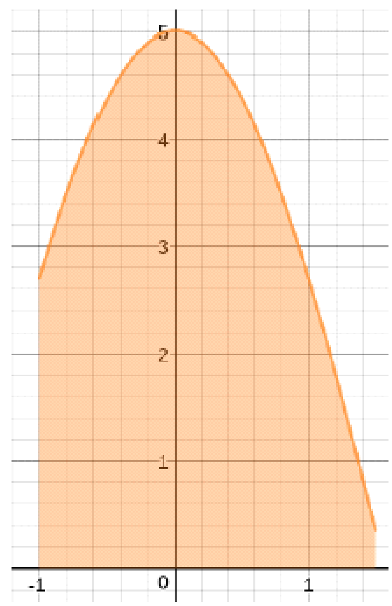
# Section 15.9: Change of Variables in Multiple Integrals

From Cal 1/Cal 2 we know the following integrals are equivalent with the substitution $u = 0.1x^2 - 1$.

$$\int_0^5 x\cos(0.1x^2 - 1)dx = \int_{-1}^{1.5} 5\cos(u)du \approx 9.19483$$



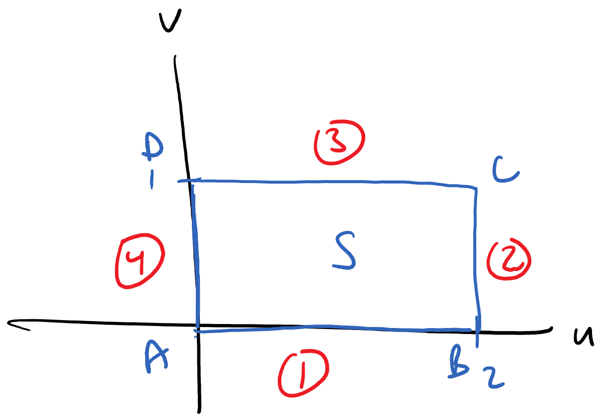
Consider $\iint\limits_R F(x,y)dA$ where $R$ is a region in the $xy-$plane. Suppose we make the substitution $x = x(u,v)$ and $y = y(u,v)$ where $x$ and $y$ are functions of $u$ and $v$ that have continuous first-order partial derivatives. These equations give a **transformation** that will take a region $S$ in the $uv-$plane and map it into a region $R$ in the $xy-$plane, also called the image of $S$.

This will give $\displaystyle\iint\limits_R F(x,y)dA = \iint\limits_S F(x(u,v), y(u,v))dA$

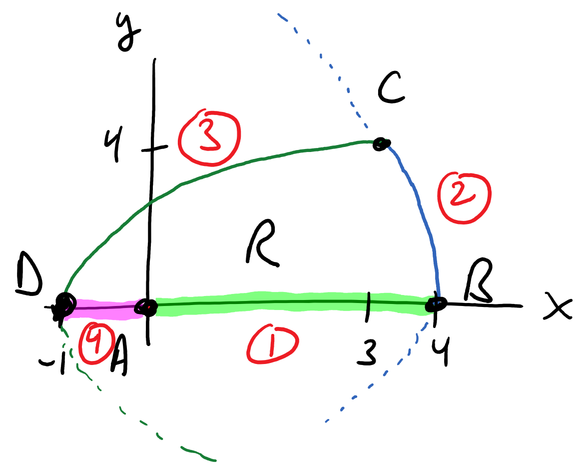
In order to find the region $S$ that transforms into region $R$, we need the transformation be one-to-one. (This means no two points $(u_1, v_1)$ and $(u_2, v_2)$ map to the same point $(x_1, y_1)$. Also needed is that as the boundary of $S$ is traversed once, then the boundary of $R$ will also be traversed only once.

Example: A transformation is defined by the equations $x = u^2 - v^2$, $y = 2uv$.

Find the image of the square $S = \{(u, v) | 0 \le u \le 2,\ 0 \le v \le 1\}$




$$\boxed{\begin{aligned} x &= u^2 - v^2 \\ y &= 2uv \end{aligned}}$$




**Corner points**

$(u, v) \longrightarrow (x, y)$

$A\ (0,0)$ $\quad$ $A\ (0,0)$

$B\ (2,0)$ $\quad$ $B\ (4,0)$

$C\ (2,1)$ $\quad$ $C\ (3,4)$

$D\ (0,1)$ $\quad$ $D\ (-1,0)$

**Side 1**

$u = u \qquad v = 0 \qquad 0 \le u \le 2$

$x = u^2 \qquad y = 0$

$0 \le x \le 4$

**Side 2**

$u = 2 \qquad 0 \le v \le 1$

$x = 4 - v^2 \qquad y = 4v \rightarrow v = \frac{1}{4}y$

$x = 4 - \left(\frac{1}{4}y\right)^2$

$$\boxed{x = 4 - \frac{y^2}{16}}$$

**Side 3**

$0 \le u \le 2 \qquad v = 1$

$x = u^2 - 1 \qquad y = 2u$

$x = \left(\frac{1}{2}y\right)^2 - 1 \qquad u = \frac{1}{2}y$

$$\boxed{x = \frac{1}{4}y^2 - 1}$$

**Side 4**

$u = 0 \qquad 0 \le v \le 1$

$x = -v^2 \qquad y = 0$

$x = -v^2 \qquad y = 0$

$-1 \le x \le 0$

## Pg 4: The Jacobian

**Definition:** The **Jacobian** of the transformation $T$ given by $x = x(u, v)$ and $y = y(u, v)$ is

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \qquad \text{or}$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$$

$X_u = 2u$

$X_v = -2v$

Example: Find the Jacobian of the transformation defined by $x = u^2 - v^2$, $y = 2uv$.

$$J = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix}$$

$$= 2u(2u) - (2v)(-2v)$$

$$J = 4u^2 + 4v^2$$

Example: Find the Jacobian of the transformation defined by $x = r\cos\theta$, $y = r\sin\theta$.

$$J = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta)$$

$$= r\cos^2\theta + r\sin^2\theta = r\left[\cos^2\theta + \sin^2\theta\right]$$

$$= r$$

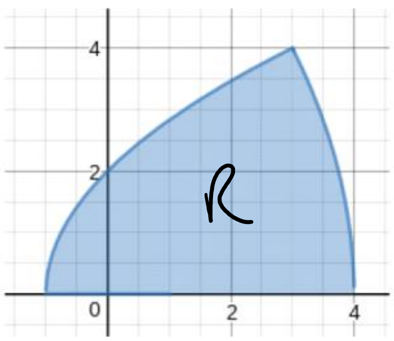
**Change of Variable in a Double Integral** Suppose $T$ is a one-to-one transformation, where the substitutions have continuous first-order partial derivatives, whose Jacobian is nonzero and that maps a region $S$ in the $uv-$plane onto a region $R$ in the $xy-$plane. Suppose that $f$ is continuous on $R$ and that $R$ and $S$ are type I or type II plane regions. Then

$$\iint\limits_R f(x,y)dA = \iint\limits_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

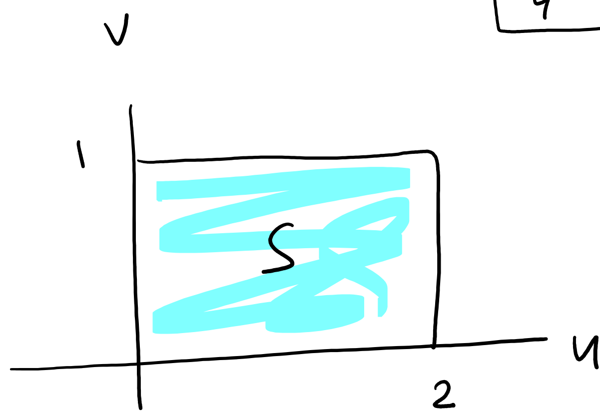
Example: Let R be the region bounded by the $x$−axis and the parabolas $y^2 = 64 - 16x$ and $y^2 = 4 + 4x$, $y \geq 0$. Use the change of variables $x = u^2 - v^2$ and $y = 2uv$ to evaluate

$$\iint_R y\, dA$$

$$y^2 = 4 + 4x$$
$$y^2 - 4 = 4x$$
$$\frac{1}{4}y^2 - 1 = x$$



Region R




$$0 \leq u \leq 2$$
$$0 \leq v \leq 1$$

$$J = 4u^2 + 4v^2$$

$$\iint_R y\, dA = \iint_S 2uv \cdot \left|4u^2 + 4v^2\right| dA$$

$$= \int_{v=0}^{1} \int_{u=0}^{2} 2uv\left(4u^2 + 4v^2\right) du\, dv$$

$$= \int_{v=0}^{1} \int_{u=0}^{2} 8u^3 v + 8uv^3 \; du\, dv$$

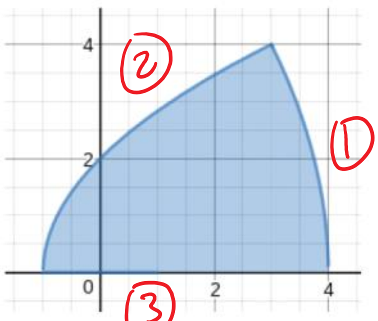
$$= \int^{1} \left(2u^4 v + 4u^2 v^3\right)\Big|^{2} dv$$

$$= \int_{v=0}^{1} \left(2u^4 v + 4u^2 v^3\right)\Big|_0^{\,} dv$$

$$= \int_{v=0}^{1} 32v + 16v^3 \, dv = 16v^2 + 4v^4 \Big|_0^1$$

$$= 16 + 4 - 0 = \underline{20}$$

Pg 8: part b



Region R

$x = u^2 - v^2$

$y = 2uv$

1

$y^2 = 64 - 16x$

$4u^2v^2 = 64 - 16(u^2 - v^2)$

$4u^2v^2 = 64 - 16u^2 + 16v^2$

$4u^2v^2 + 16u^2 = 64 + 16v^2$

$u^2v^2 + 4u^2 = 16 + 4v^2$

$u^2v^2 + 4u^2 - (16 + 4v^2) = 0$

$u^2(v^2 + 4) - 4(4 + v^2) = 0$

$(u^2 - 4)(v^2 + 4) = 0$

$u^2 = 4 \qquad \underbrace{v^2 + 4 = 0}_{\text{not possible}}$

$u = \pm 2$

2

$y^2 = 4 + 4x$

$4u^2v^2 = 4 + 4u^2 - 4v^2$

$4u^2v^2 - 4u^2 = 4 - 4v^2$

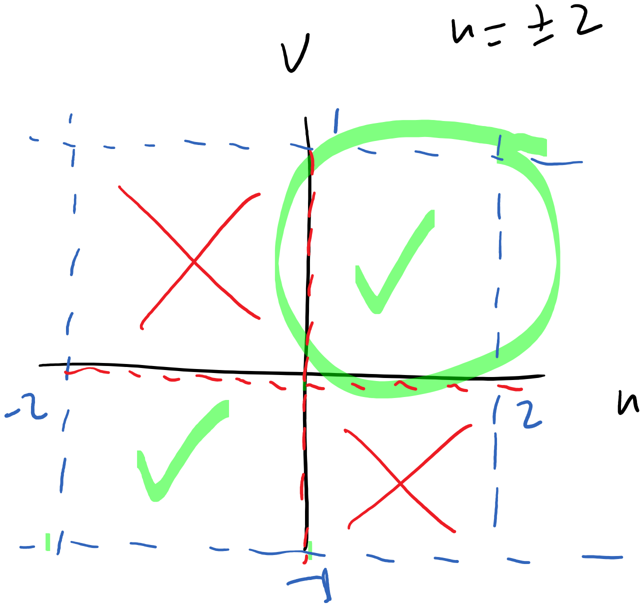
$4u^2(v^2 - 1) = -4(v^2 - 1)$

$4u^2(v^2 - 1) + 4(v^2 - 1) = 0$

$(4u^2 + 4)(v^2 - 1) = 0$

$\underbrace{4u^2 + 4 = 0}_{\text{not possible}} \qquad v^2 = 1$

$v = \pm 1$




3 $\quad y = 0$

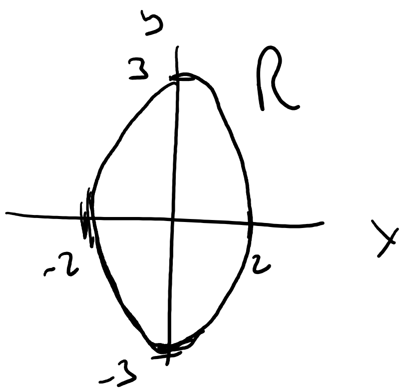
$2uv = 0$

$u = 0 \quad$ or $v = 0$

$\left.\begin{aligned} x &= u^2 - v^2 \\ y &= 2uv \end{aligned}\right] \quad y \geq 0$

Example: Let R be the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$. Evaluate by making the given transformation.

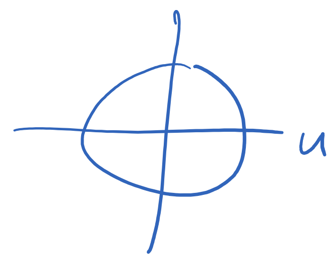
$\iint_R y+3\;dA$ with $x=2u$ and $y=3v$




$$\frac{(2u)^2}{4}+\frac{(3v)^2}{9}=1$$

$$\frac{4u^2}{4}+\frac{9v^2}{9}=1$$

$$u^2+v^2=1$$




$x=0 \qquad \frac{y^2}{9}=1$

$y^2=9$

$$J=\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}=\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}$$

$$J=2(3)-0=6$$

$$|J|=|6|=6$$

polar

$u=r\cos\theta$

$v=r\sin\theta$

$0\le r\le 1$

$0\le\theta\le 2\pi$

$$\iint_R y+3\;dA=\iint_S (3v+3)\,6\cdot dA$$

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{1}(18r\sin\theta+18)\,r\,dr\,d\theta$$

$$\int^{2\pi}\int^{1} \quad dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 18r^2 \sin\theta + 18r \, dr \, d\theta$$

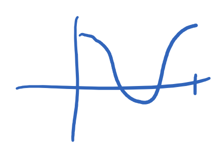
$$= \int_{\theta=0}^{2\pi} \left(6r^3 \sin\theta + 9r^2\right)\Big|_0^1 \, d\theta$$

$$= \int_0^{2\pi} 6\sin\theta + 9 \, d\theta = \left(-6\cos\theta + 9\theta\right)\Big|_0^{2\pi}$$

$$= -6\cos(2\pi) + 18\pi - \left(-6\cos(0) + 0\right)$$

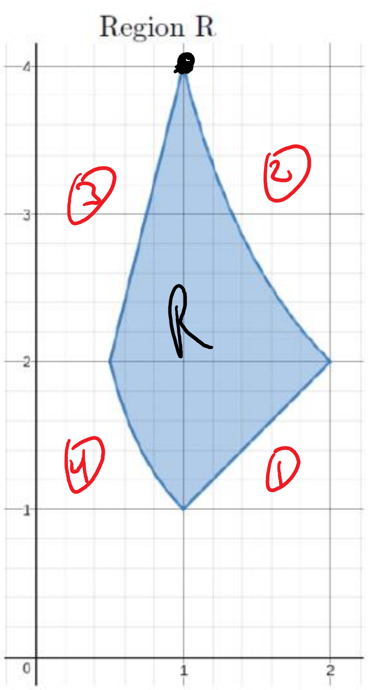
$$= -6 + 18\pi + 6$$

$$= 18\pi$$



Example: Let R be the region in the first quadrant bounded by the lines $y = x$ and $y = 4x$ and the hyperbolas $xy = 1$ and $xy = 4$. Evaluate by making the given transformation.

$\iint_R xy\, dA$ with $x = u/v$ and $y = v$



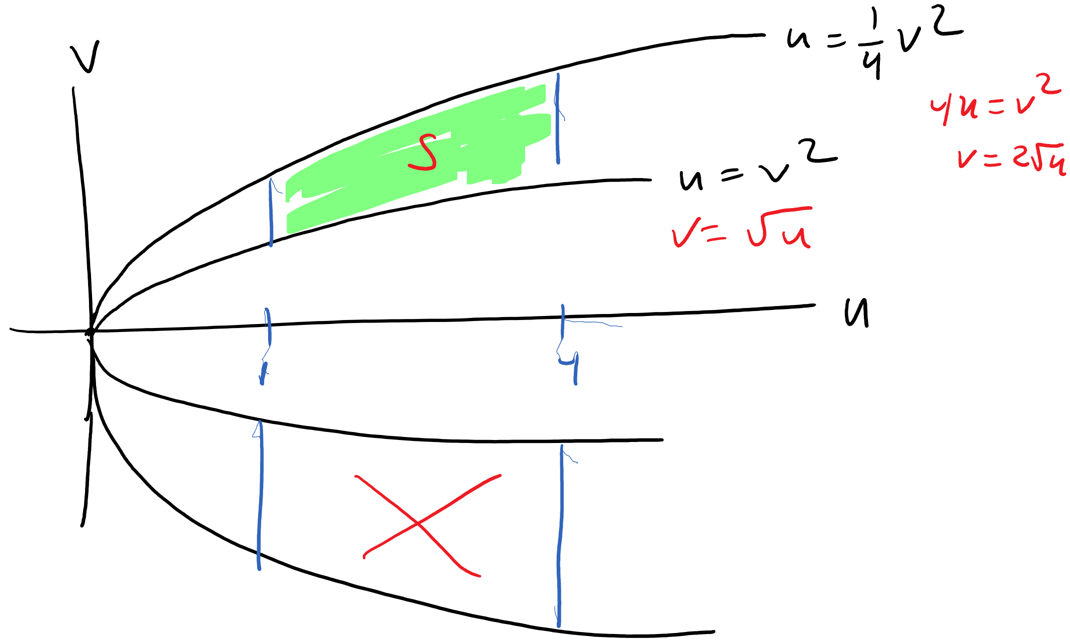

① $y = x \longrightarrow v = \frac{u}{v} \longrightarrow v^2 = u$

③ $y = 4x \longrightarrow v = \frac{4u}{v} \longrightarrow v^2 = 4u$

$u = \frac{1}{4} v^2$

② $xy = 4 \longrightarrow \frac{u}{v} \cdot v = 4 \longrightarrow u = 4$

④ $xy = 1 \longrightarrow \frac{u}{v} \cdot v = 1 \longrightarrow u = 1$




$$\boxed{\begin{gathered} 1 \le u \le 4 \\ \sqrt{u} \le v \le 2\sqrt{u} \end{gathered}}$$

$x = \frac{u}{v} \qquad y = v$

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}(1) - \frac{-u}{v^2}\cdot 0$$

$$J = \frac{1}{v}$$

$$|J| = \left|\frac{1}{v}\right| = \frac{1}{v}$$

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$$\iint_R xy\, dA = \iint_S \frac{u}{v}\cdot v \cdot \frac{1}{v}\, dA$$

$$= \int_{u=1}^{4} \int_{v=\sqrt{u}}^{2\sqrt{u}} \frac{u}{v}\, dv\, du = \int_{u=1}^{4} u\ln|v| \Big|_{\sqrt{u}}^{2\sqrt{u}} du$$

$$= \int_{u=1}^{4} u\ln(2\sqrt{u}) - u\ln(\sqrt{u}) \; du$$

$$= \int_{u=1}^{4} u\left[\ln(2) + \ln(\sqrt{u})\right] - u\ln(\sqrt{u}) \; du$$

$$\int^{4} \quad u\ln\sqrt{u} - u\ln\sqrt{u} \quad du$$

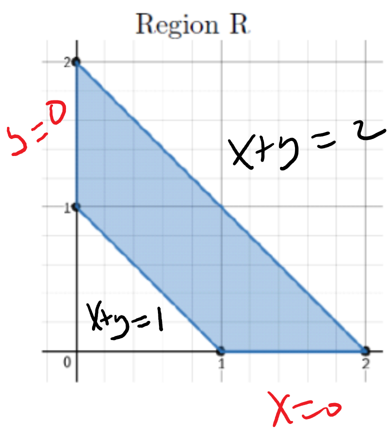
$$= \int_{u=1}^{4} u\ln(2) + u\ln\sqrt{u} - u\ln\sqrt{u} \; du$$

$$= \int_{u=1}^{4} u\ln(2)\; du = \frac{1}{2}u^2\ln(2)\Big|_1^4$$

$$= 8\ln(2) - \frac{1}{2}\ln(2)$$

Example: Let R be the region in the $xy$−plane bounded by the vertices $(0,1)$, $(0,2)$, $(2,0)$, and $(1,0)$. Evaluate

$$\iint_R e^{(y-x)/(y+x)}\,dA$$




$$u = y - x$$
$$v = y + x$$

add the eq.

$$u + v = 2y$$
$$y = \frac{1}{2}(u+v)$$

subtract the eq

$$u - v = -2x$$
$$-\frac{1}{2}(u-v) = x$$
$$x = \frac{1}{2}v - \frac{1}{2}u$$

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$|J| = \left|-\frac{1}{2}\right| = \frac{1}{2}$$

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$$x + y = 2 \longrightarrow v = 2$$

$$x + y = 1 \longrightarrow v = 1$$

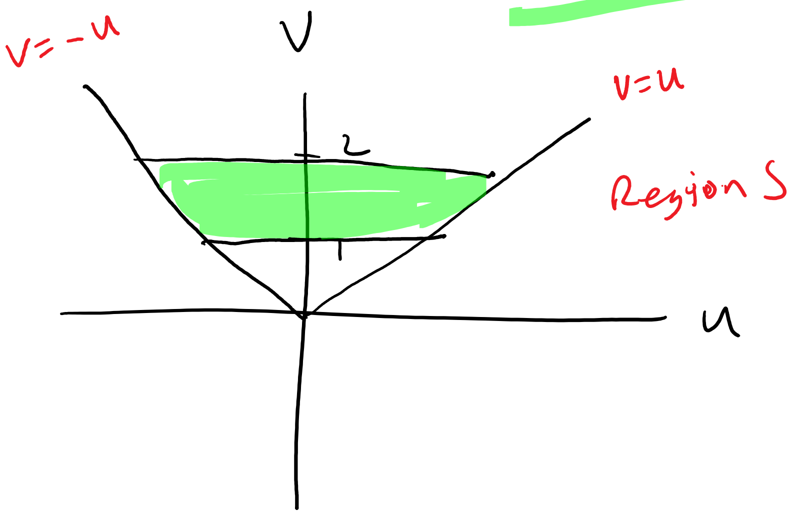
$$x = 0 \longrightarrow \frac{1}{2}v - \frac{1}{2}u = 0$$
$$\frac{1}{2}v = \frac{1}{2}u$$
$$v = u$$

$$x = \frac{1}{2}v - \frac{1}{2}u$$
$$y = \frac{1}{2}u + \frac{1}{2}v$$

$$y=0 \longrightarrow 0 = \frac{1}{2}u + \frac{1}{2}v$$

$$-\frac{1}{2}u = \frac{1}{2}v$$

$$v = -u$$




$$1 \le v \le 2$$

$$-v \le u \le v$$

$$\iint_R e^{\frac{y-x}{y+x}}\, dA = \int_{v=1}^{2}\int_{u=-v}^{v} e^{\frac{u}{v}} \cdot \frac{1}{2}\, du\, dv$$

$$= \frac{1}{2}\int_{v=1}^{2} \frac{1}{\frac{1}{v}} e^{\frac{u}{v}} \Big|_{u=-v}^{v}\, dv \qquad e^{3u} \to \frac{1}{3}e^{3u}$$

$$= \frac{1}{2}\int_{v=1}^{2} v e^{\frac{u}{v}} \Big|_{u=-v}^{v}\, dv = \frac{1}{2}\int_{v=1}^{2} ve^{1} - ve^{-1}\, dv$$

$$= \frac{1}{2}\int^{2} v \cdot (e^{1} - e^{-1})\, dv$$

$$= \frac{1}{2}\int_{v=1}^{2} v\cdot\left(e^{1}-e^{-1}\right)\,dv$$

$$= \frac{1}{2}\cdot\frac{1}{2}v^{2}\left(e^{1}-e^{-1}\right)\Big|_{v=1}^{2}$$

$$= \frac{1}{4}\cdot 4\cdot\left(e^{1}-e^{-1}\right) - \frac{1}{4}\cdot 1\left(e^{1}-e^{-1}\right)$$

$$= \underline{\frac{3}{4}\left(e^{1}-e^{-1}\right)}$$

## Triple Integrals

Given the transformation $x = x(u, v, w)$, $y = y(u, v, w)$ and $z = z(u, v, w)$ then the Jacobian is the following $3 \times 3$ determinant.

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

With a hypotheses similar to the double integral change of variables we have the following for the triple integral.

$$\iiint\limits_R f(x, y)\,dV = \iiint\limits_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\,dv\,dw$$

Example: Find the Jacobian for the transformation.

$$x = \rho \sin\phi \ \cos\theta \qquad y = \rho \sin\phi \ \sin\theta \qquad z = \rho \cos\phi$$

We compute the Jacobian as follows:

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin\phi \ \cos\theta & -\rho \sin\phi \ \sin\theta & \rho \cos\phi \ \cos\theta \\ \sin\phi \ \sin\theta & \rho \sin\phi \ \cos\theta & \rho \cos\phi \ \sin\theta \\ \cos\phi & 0 & -\rho \sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho \sin\phi \ \sin\theta & \rho \cos\phi \ \cos\theta \\ \rho \sin\phi \ \cos\theta & \rho \cos\phi \ \sin\theta \end{vmatrix} - \rho \sin\phi \begin{vmatrix} \sin\phi \ \cos\theta & -\rho \sin\phi \ \sin\theta \\ \sin\phi \ \sin\theta & \rho \sin\phi \ \cos\theta \end{vmatrix}$$

$$= \cos\phi \, (-\rho^2 \sin\phi \ \cos\phi \ \sin^2\theta - \rho^2 \sin\phi \ \cos\phi \ \cos^2\theta)$$

$$- \rho \sin\phi \, (\rho \sin^2\phi \ \cos^2\theta + \rho \sin^2\phi \ \sin^2\theta)$$

$$= -\rho^2 \sin\phi \ \cos^2\phi - \rho^2 \sin\phi \ \sin^2\phi = \underline{-\rho^2 \sin\phi}$$

Since $0 \leqslant \phi \leqslant \pi$, we have $\sin\phi \geqslant 0$. Therefore

$$\left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} \right| = |-\rho^2 \sin\phi| = \rho^2 \sin\phi$$