Section 15.6: Triple Integrals

Let B be a rectangular box such that $B = \{(x, y, z) | a \le x \le b, c \le y \le d, r \le z \le s\}$, i.e. $B = \underbrace{[a, b]}_{x} \times \underbrace{[c, d]}_{y} \times \underbrace{[r, s]}_{z}$.

Definition: The **triple integral** of $f(x, y, z)$ over the box B is

$$\iiint_B f(x, y, z)\,dV = \lim_{||P|| \to 0} \sum_i \sum_j \sum_k f(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}$$

if this limit exists.

Note: The volume of solid E is given by $\iiint_E 1 \; dV$.

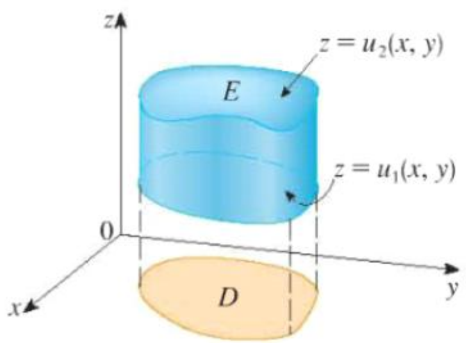
Fubini's Theorem for Triple Integrals: If f is continuous on the rectangular box
$B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\,dV = \int_{z=r}^{s} \int_{y=c}^{d} \int_{x=a}^{b} f(x, y, z) \; \underline{dx\,dy\,dz} = \int_{x=a}^{b} \int_{z=r}^{s} \int_{y=c}^{d} f(x, y, z) \; \underline{dy\,dz\,dx}$$

Triple Integral over General Regions:

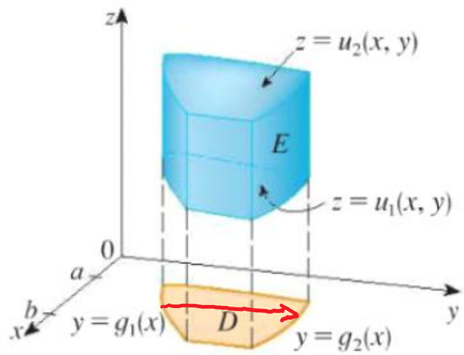
When using a non-rectangular solid, we consider the projection (image) the solid makes on the different coordinate planes.

A **Type I** region is the solid $E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$



Top

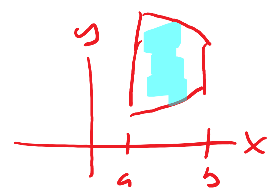
Bottom



Top

Bottom

dy dx



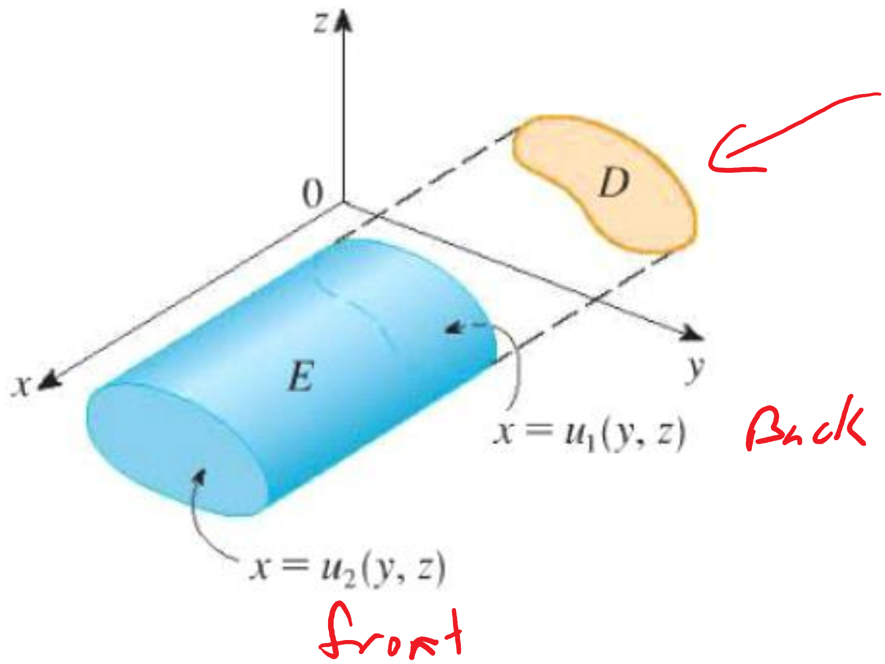
$$\boxed{\begin{array}{c} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{array}}$$

Top

$$\iiint\limits_E f(x, y, z)\,dV = \iint\limits_D \underbrace{\left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,dz\right]}_{\text{Bottom}} dA$$

$$= \int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} \int_{z=u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,dz\,dy\,dx$$

A **Type 2** region is the solid $E = \{(x, y, z) | (y, z) \in D, u_1(y, z) \le x \le u_2(y, z)\}$

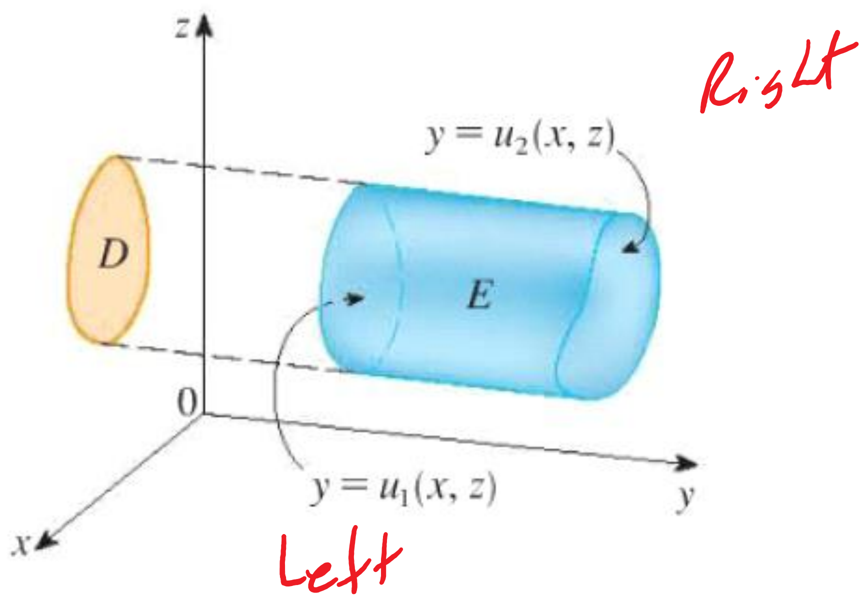


front.

$$\iiint_E f(x, y, z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\,\underline{\underline{dx}} \right] dA$$

Back

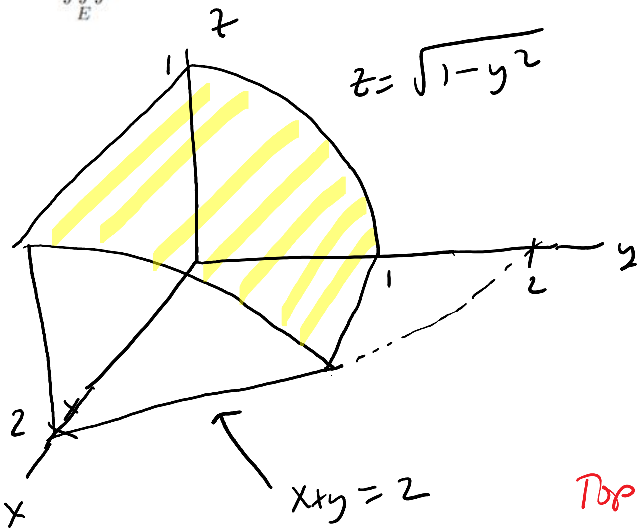
A **Type 3** region is the solid $E = \{(x, y, z) | (x, z) \in D, u_1(x, z) \le y \le u_2(x, z)\}$



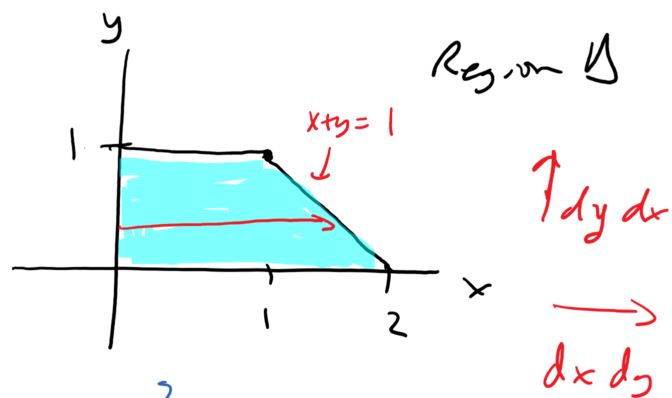
$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{\overset{u_1(x,z)}{\text{left}}}^{\overset{\text{Right}}{u_2(x,z)}} f(x, y, z)\underline{\underline{dy}} \right] dA$$

Example: Given E is the solid bounded by the plane $x + y = 2$ and the cylinder $y^2 + z^2 = 1$ in the first octant. Setup the triple integral with different projections on the different coordinate planes.

$$\iiint_E z\, dV$$



method 1 project on xyplane



Top $z = \sqrt{1-y^2}$

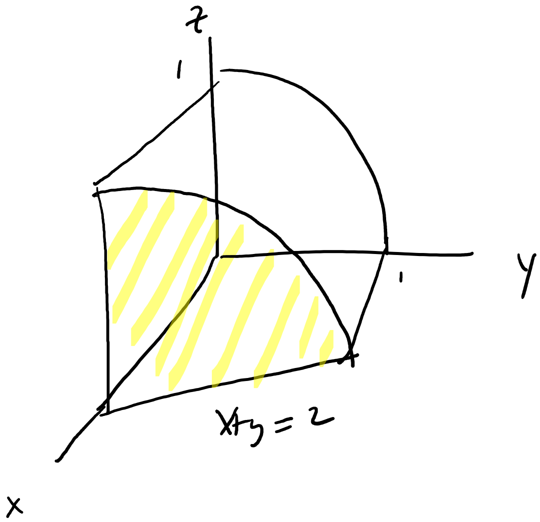
Bottom $z = 0$

$0 \le y \le 1$

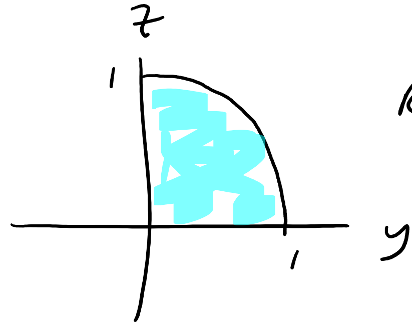
$0 \le x \le 1-y$

$$\iiint_E z\, dV = \iint_D \left[\int_{z=0}^{\sqrt{1-y^2}} z\, dz \right] dA$$

$$= \underbrace{\int_{y=0}^{1} \int_{x=0}^{1-y} \int_{z=0}^{\sqrt{1-y^2}} z\, dz\, dx\, dy}= \dots = \frac{13}{24}$$



method 2: project onto yz plane



Region D.

front $x = 2 - y$

Back $x = 0$

$0 \le y \le 1$

$0 \le z \le \sqrt{1-y^2}$

$$\boxed{\iiint_E z \, dV = \int_{y=0}^{1} \int_{z=0}^{\sqrt{1-y^2}} \int_{x=0}^{2-y} z \, dx \, dz \, dy}$$

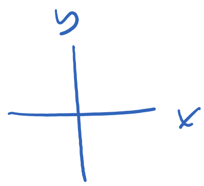
$$= \int_{y=0}^{1} \int_{z=0}^{\sqrt{1-y^2}} zx \Big|_{x=0}^{2-y} dz \, dy$$

$$= \int^{1} \int^{\sqrt{1-y^2}} z(2-y) \, dz \, dy$$

now

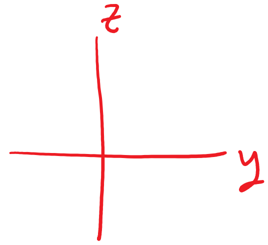
$$\int_{y=0} \int_{z=0} z(2-y)\, dz \cdot d$$

now convert to polar



$x = r\cos\theta$

$y = r\sin\theta$



$y = r\cos\theta$

$z = r\sin\theta$

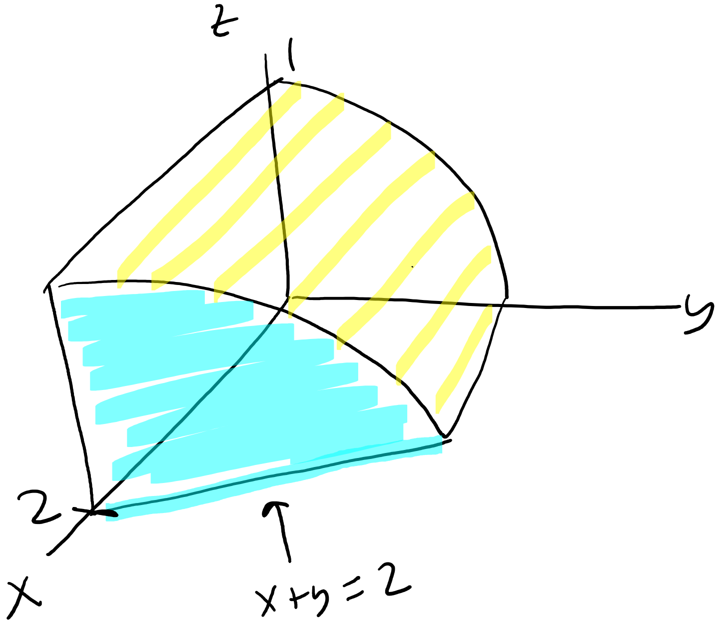
$0 \le \theta \le \pi/2$

$0 \le r \le 1$

$$\int_{\theta=0}^{\pi/2} \int_{r=0}^{1} r\sin\theta\,(2 - r\cos\theta) \cdot r\, dr\, d\theta$$

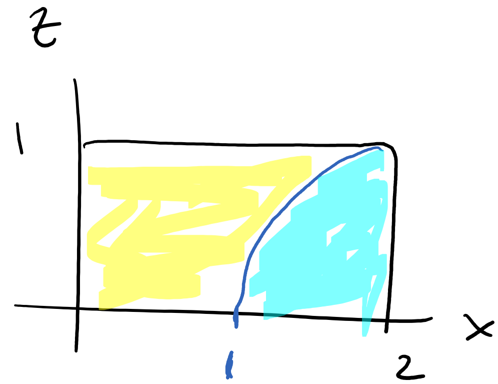
$$= \cdots = \frac{13}{24}$$

Pg 5c



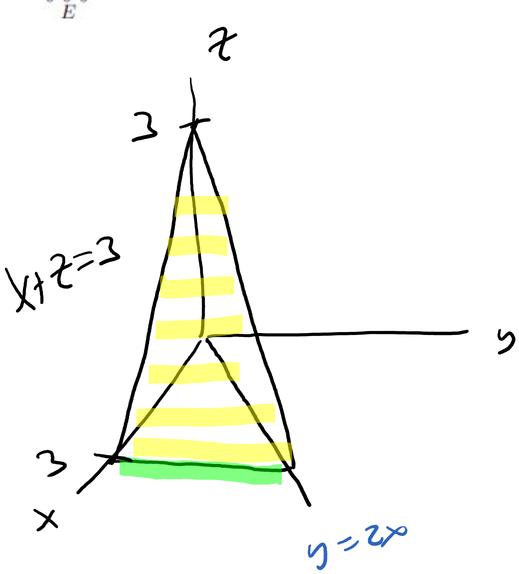
method 3

projection on xz plane.



Example: Given E is the solid bounded by the planes $\underbrace{z = 3 - x}_{x+z=3}$, $z = 0$, $y = 0$, and $y = 2x$.

Rewrite $\displaystyle\iiint_E f(x, y, z)\, dV$ as 6 different interated integrals.

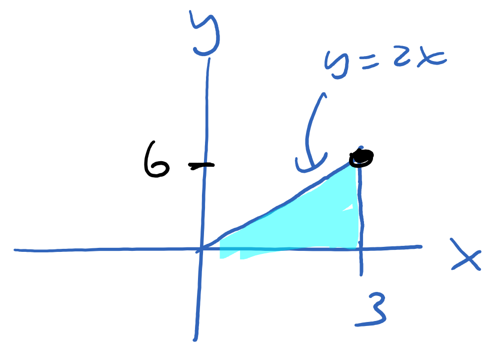


method project on xy plane.

Top $z = 3 - x$

Bottom $z = 0$

Region D



$dy\,dx$ ↑

$0 \le x \le 3$

$0 \le y \le 2x$

$dx\,dy$ →

$0 \le y \le 6$

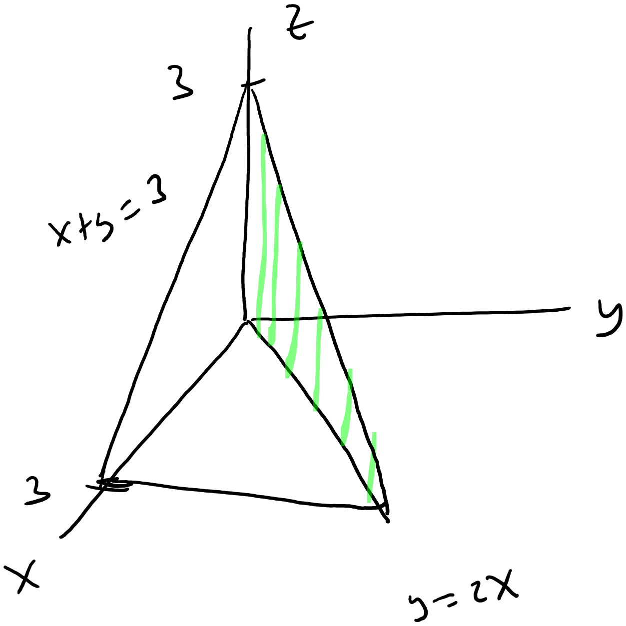
$\frac{y}{2} \le x \le 3$

$$\int_{x=0}^{3} \int_{y=0}^{2x} \int_{z=0}^{3-x} f(x,y,z)\, dz\, dy\, dx$$

$$\int_{y=0}^{6} \int_{x=\frac{y}{2}}^{3} \int_{z=0}^{3-x} f(x,y,z)\, dz\, dx\, dy$$

$y = 0$, $x = \frac{y}{2}$, $z = 0$

Pg 6 cont.

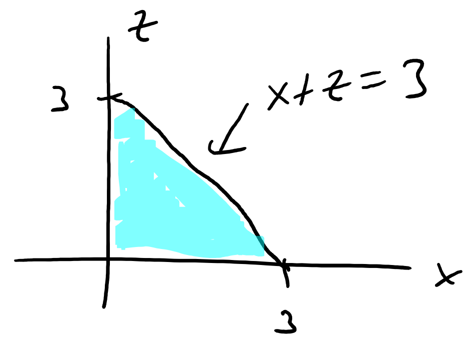


method: project on xz plane

Left $y=0$

Right $y=2x$

Region D



$dz\,dx$ ↑

$0 \le x \le 3$

$0 \le z \le 3-x$

$dx\,dz$ →

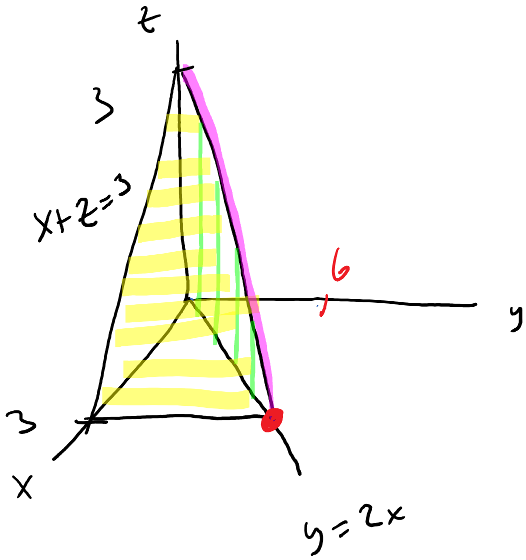
$0 \le z \le 3$

$0 \le x \le 3-z$

$$\int_{x=0}^{3}\int_{z=0}^{3-x}\int_{y=0}^{2x} f(x,y,z)\,dy\,dz\,dx$$

$$\int_{z=0}^{3}\int_{x=0}^{3-z}\int_{y=0}^{2x} f(x,y,z)\,dy\,dx\,dx$$

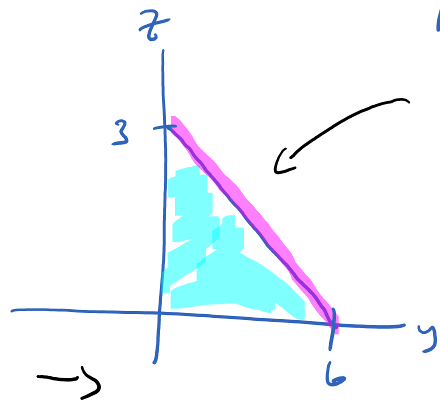
Pg 6 cont.



Method project on yz plane

front $x = 3 - z$

Back $x = \frac{y}{2}$



Region D

This is the intersection of 2 planes

$x + z = 3$
$y = 2x$

$x = \frac{1}{2}y$

$dz\,dy$ ↑

$0 \le y \le 6$
$0 \le z \le 3 - \frac{1}{2}y$

$dy\,dz$ →

$0 \le z \le 3$
$0 \le y \le 3 - 2z$

$\frac{1}{2}y + z = 3$
$z = 3 - \frac{1}{2}y$

$y + 2z = 3$
$y = 3 - 2z$

$$\int_{y=0}^{6} \int_{z=0}^{3-\frac{1}{2}y} \int_{x=\frac{y}{2}}^{3-z} f(x,y,z)\,dx\,dz\,dy$$

$$\int_{z=0}^{3} \int_{y=0}^{3-2z} \int_{x=\frac{y}{2}}^{3-z} f(x,y,z)\,dx\,dy\,dz$$

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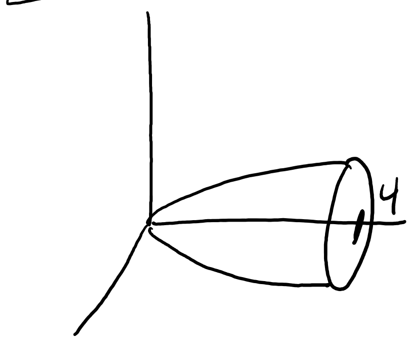
$z^2 = y - x^2$

Example: Given E is the solid bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$. Compute

$$\iiint_E \sqrt{x^2+z^2}\, dV$$



projection xy plane.

Top $z = \sqrt{y - x^2}$

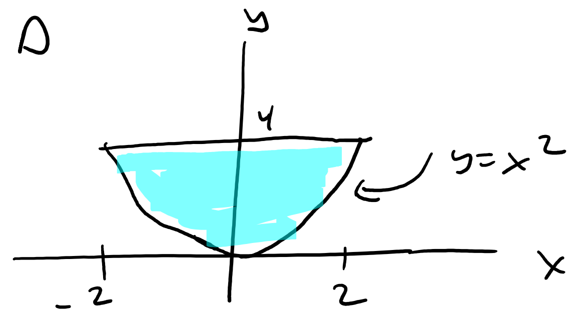
Bottom $z = -\sqrt{y - x^2}$

Let $z = 0$

$y = x^2 + z^2$

$y = x^2$

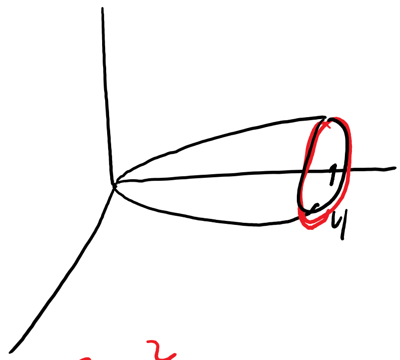
Region D



$-2 \le x \le 2$

$x^2 \le y \le 4$

$$\iiint_E \sqrt{x^2+z^2}\, dV = \int_{x=-2}^{2} \int_{y=x^2}^{4} \int_{z=-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\, dz\, dy\, dx$$



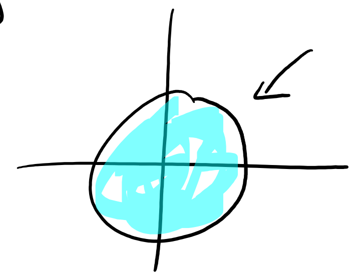
$y = x^2 + z^2$

project on xz plane.

Left: $y = x^2 + z^2$

Right: $y = 4$

Region D



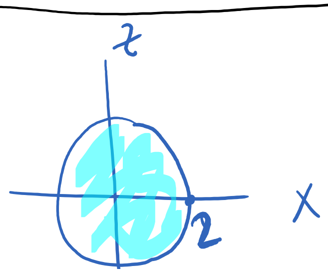
$x^2 + z^2 = 4$

$$\iiint_E \sqrt{x^2+z^2}\, dV = \iint_D \left[\int_{y=x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy \right] dA$$

$$= \iint_D y\sqrt{x^2+z^2} \Big|_{y=x^2+z^2}^{4} \, dA$$

$$= \iint 4\sqrt{x^2+z^2} - (x^2+z^2)\sqrt{x^2+z^2} \quad dA$$

$$= \iint_D \left[4 - (x^2+z^2)\right] \sqrt{x^2+z^2} \quad dA$$



$x = r\cos\theta$

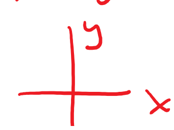
$z = r\sin\theta$

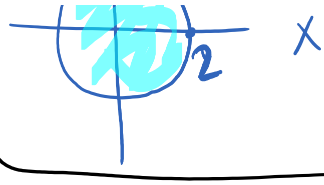
$x^2 + z^2 = r^2$

$x = r\cos\theta$

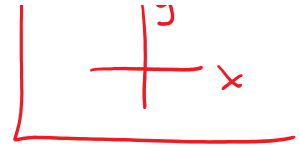
$y = r\sin\theta$

$x^2 + y^2 = r^2$





$$x^2+z^2=r^2$$



$0 \le \theta \le 2\pi$

$0 \le r \le 2$

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{2}(4-r^2)\sqrt{r^2}\cdot r\,dr\,d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} 4r^2-r^4\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} d\theta \cdot \int_{r=0}^{2} 4r^2-r^4\,dr$$

$$= \theta\Big|_0^{2\pi} \cdot \left(\frac{4r^3}{3}-\frac{r^5}{5}\right)\Big|_0^{2}$$

$$= 2\pi \cdot \left(\frac{32}{3}-\frac{32}{5}\right) = \frac{128\pi}{15}$$

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Applications of Triple Integrals:
Given a solid E with density function $\rho(x, y, z)$.

mass: $m = \iiint\limits_E \rho(x, y, z)\, dV$

moments about the yz coordinate plane: $M_{yz} = \iiint\limits_E x\, \rho(x, y, z)\, dV$

moments about the xz coordinate plane: $M_{xz} = \iiint\limits_E y\, \rho(x, y, z)\, dV$

moments about the xy coordinate plane: $M_{xy} = \iiint\limits_E z\, \rho(x, y, z)\, dV$

center of Mass: $(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right)$