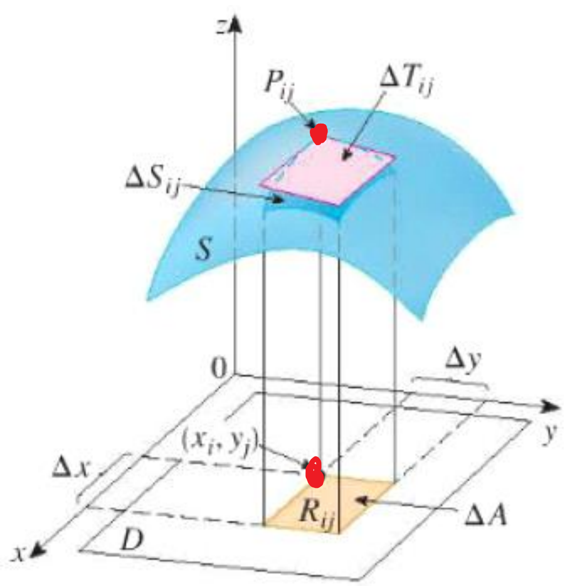
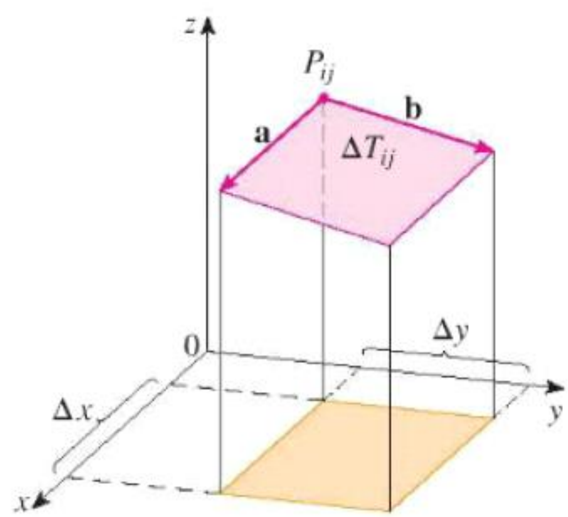
Section 15.5: Surface Area

Let S be a surface with equation $z = f(x, y)$. Assume that this surface is above the xy-plane and the domain D of f is a rectangular region. Let R_{ij} be a rectangular sub-partition of D where (x_i, y_j) is the corner of R_{ij} that is closest to the origin.

Notice from the figure, that the section of tangent plane, ΔT_{ij} at the point $P_{ij}(x_i, y_j, f(x_i, y_j))$ over the region R_{ij} will approximate the surface area on that region of the domain. Thus $A(S) \approx \sum_{i=1} \sum_{j=1} \Delta T_{ij}$





Let $\mathbf{a}$ and $\mathbf{b}$ be vectors that start at point P_{ij} and lie along the edge of ΔT_{ij}.

Thus $\mathbf{a} = \langle \Delta x, 0, f_x(x_i, y_i)\Delta x\rangle$ and $\mathbf{b} = \langle 0, \Delta y, f_y(x_i, y_i)\Delta y\rangle$ and the area of $\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}|$.

Now $\mathbf{a} \times \mathbf{b} = \langle -f_x(x_i, y_j)\Delta x\Delta y, \;\; -f_y(x_i, y_j)\Delta x\Delta y, \;\; \Delta x\Delta y\rangle$ Since $\Delta x\Delta y =$ ΔA ΔA we get

$\mathbf{a} \times \mathbf{b} = \langle -f_x(x_i, y_j)\Delta A, \;\; -f_y(x_i, y_j)\Delta A, \;\; \Delta A\rangle$ which gives

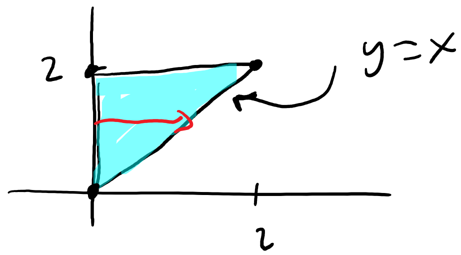
$$\Delta T_{ij} = \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1} \;\; \Delta A$$

and $A(S) \approx \sum_{i=1}\sum_{j=1} \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1} \;\; \Delta A$

Definition: The area of the surface with equation $z = f(x, y)$ over the region D where f_x and f_y are continuous is given by

$$A(S) = \iint_D \sqrt{[f_x]^2 + [f_y]^2 + 1}\, dA$$

Example: Find the surface area of the part of the surface $z = 3x + y^2$ that lies above the triangle region in the xy-plane with vertices $(0,0)$, $(0,2)$, and $(2,2)$.



$z_x = 3$

$z_y = 2y$

$dx\,dy$

$0 \leq y \leq 2$

$0 \leq x \leq y$

$$SA = \iint_D \sqrt{(3)^2 + (2y)^2 + 1}\, dA$$

$$= \iint_D \sqrt{10 + 4y^2}\, dA$$

$$= \int_{y=0}^{2} \int_{x=0}^{y} \sqrt{10 + 4y^2}\, dx\, dy$$

$$= \int_{y=0}^{2} x\sqrt{10 + 4y^2}\,\Big|_0^y\, dy$$

$$= \int_0^2 y\sqrt{10 + 4y^2}\, dy$$

$u = 10 + 4y^2$

$du = 8y\, dy$

$\frac{1}{8}du = y\,dy$

$y = 0 \rightarrow u = 10$

$$= \int^{26}$$

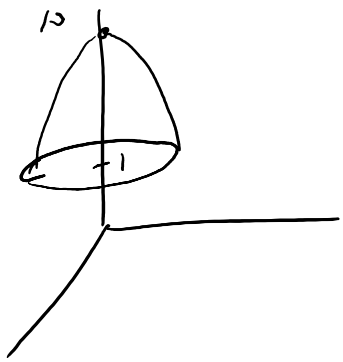
$$= \int_{10}^{26} \frac{1}{8}\sqrt{u}\,du$$

$y=0 \rightarrow u=10$

$y=2 \rightarrow u=26$

$$= \frac{1}{8}\,\frac{2}{3}u^{3/2}\Big|_{10}^{26} = \boxed{\frac{1}{12}\left[26^{3/2} - 10^{3/2}\right]}$$

Example: Find the surface area of the paraboloid given by $z = 10 - x^2 - y^2$ for $z \geq 1$.

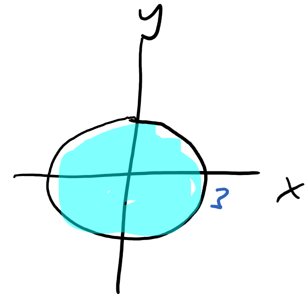


$z_x = -2x$

$z_y = -2y$

$1 = 10 - x^2 - y^2$

$x^2 + y^2 = 9$



use polar

$0 \leq \theta \leq 2\pi$

$0 \leq r \leq 3$

$$SA = \iint_D \sqrt{(-2x)^2 + (-2y)^2 + 1}\, dA$$

$$= \iint_D \sqrt{4x^2 + 4y^2 + 1}\, dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{3} \sqrt{4r^2+1} \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} \underbrace{\int_0^3 r\sqrt{4r^2+1}\, dr}\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^3 r\sqrt{4r^2+1}\, dr$$

$u = 4r^2 + 1$

$du = 8r\, dr$

$\frac{1}{8}\, du = r\, dr$

$r = 0 \quad u = 1$

$$= 2\pi \cdot \int_{u=1}^{37} \frac{1}{8}\sqrt{u}\, du$$

$u = 1$

$r = 0 \quad u = 1$

$r = 3 \quad u = 37$

$$= 2\pi \cdot \frac{1}{8} \cdot \frac{2}{3} u^{3/2} \Big|_1^{37}$$

$$= 2\pi \cdot \frac{1}{12} \left[37^{3/2} - 1 \right]$$

$$= \boxed{\frac{\pi}{6} \left[37^{3/2} - 1 \right]}$$