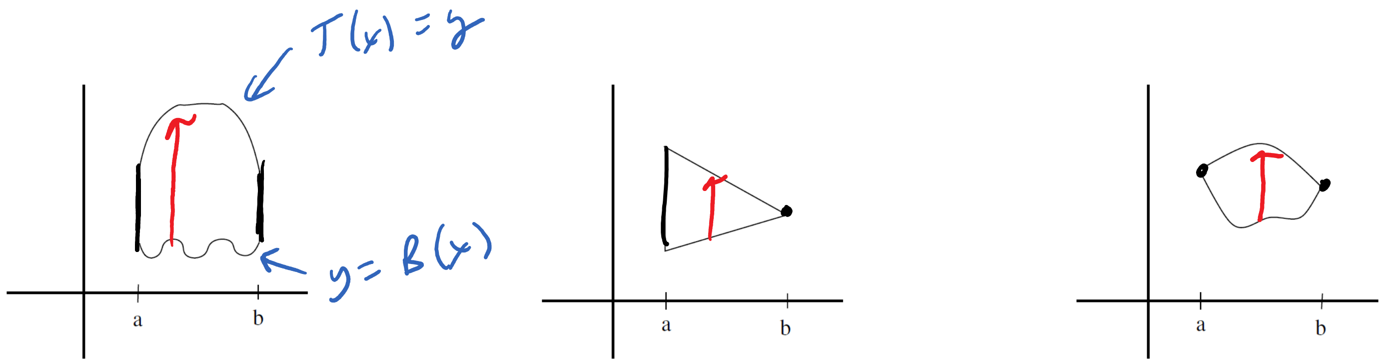
Section 15.2: Double Integrals over General Regions

Definition: A plane region D is said to be of **Type I** if it lies between two continuous functions of x, that is

$$D = \{(x, y) |\ a \le x \le b,\ B(x) \le y \le T(x)\}$$



Theorem: If f is continuous on a type I region D such that $D = \{(x, y) |\ a \le x \le b, B(x) \le y \le T(x)\}$ then

dy dx

$$\iint_D f(x, y)dA = \int_{x=a}^{b} \int_{y=B(x)}^{T(x)} f(x, y)\ dydx$$

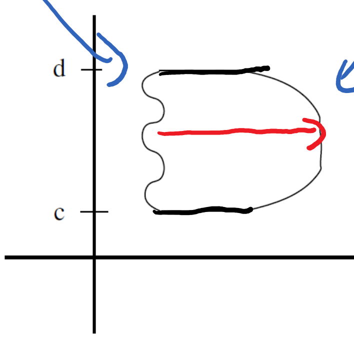
Definition: A plane region D is said to be of **Type II** if it lies between two continuous functions of y, that is

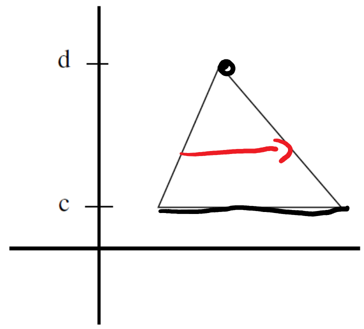
$$D = \{(x, y)|\ c \le y \le d,\ L(y) \le x \le R(y)\}$$

$dx\, dy$

$x = L(y)$

$R(y) = x$



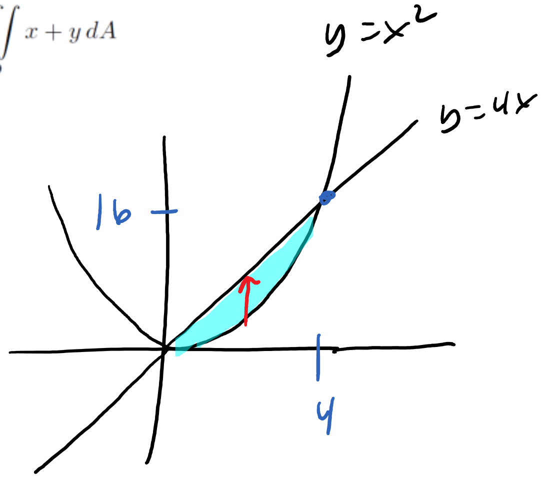


Theorem: If f is continuous on a type II region D such that $D = \{(x, y)|\ c \le y \le d, L(y) \le x \le R(y)\}$then

$$\iint\limits_D f(x, y)dA = \int\limits_{y=c}^{d} \int\limits_{x=L(y)}^{R(y)} f(x, y)\ dxdy$$

Example: If D is the region bounded by $y = 4x$ and $y = x^2$ evaluate

$$\iint_D x + y \, dA$$



find Intersection

$4x = x^2$

$0 = x^2 - 4x$

$0 = x(x-4)$

$x = 0 \quad x = 4$

$dy\,dx \qquad \overrightarrow{dx\,dy}$

Type 1 Region

$0 \le x \le 4 \qquad x^2 \le y \le 4x$

$$\iint_D x+y \, dA = \int_{x=0}^{4} \int_{y=x^2}^{4x} x+y \, dy\, dx = \int_{x=0}^{4} \underbrace{xy + \frac{y^2}{2} \Big|_{y=x^2}^{4x}} \, dx$$

$$= \int_{x=0}^{4} x(4x) + \frac{(4x)^2}{2} - \left[x(x^2) + \frac{(x^2)^2}{2} \right] dx$$

$$= \int_{x=0}^{4} 4x^2 + \frac{16x^2}{2} - x^3 - \frac{x^4}{2} \, dx$$

$$= \int_{x=0}^{4} 12x^2 - x^3 - \frac{1}{2}x^4 \, dx = 4x^3 - \frac{x^4}{4} - \frac{1}{10}x^5 \Big|_{x=0}^{4}$$

$$= 4(4)^3 - \frac{4^4}{4} - \frac{1}{10}(4)^5 - (0) = 89.6$$

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Pg 3 part 2

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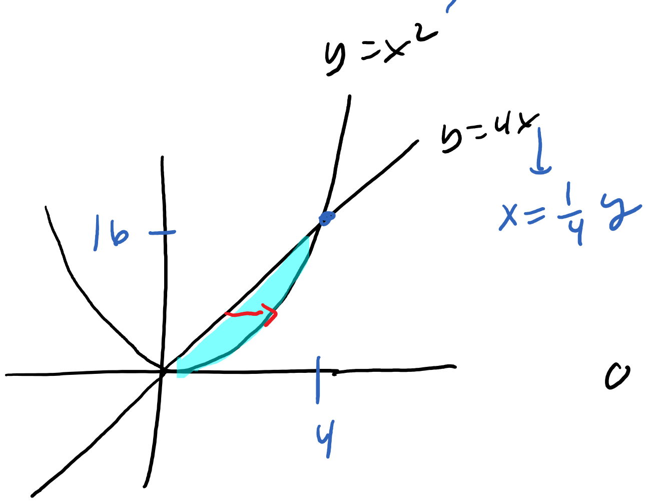
$$\iint_D x + y \, dA$$



$\uparrow dy\,dx$ $\quad \overrightarrow{dx\,dy}$

Type 2 Region

$0 \le y \le 16 \qquad \frac{1}{4}y \le x \le \sqrt{y}$

$$\iint_D x+y \, dA = \int_{y=0}^{16} \int_{x=\frac{1}{4}y}^{\sqrt{y}} x+y \, dx\, dy = \int_{y=0}^{16} \underbrace{\frac{x^2}{2} + xy \Big|_{x=\frac{1}{4}y}^{\sqrt{y}}} \, dy$$

$$= \int_{y=0}^{16} \frac{(\sqrt{y})^2}{2} + \sqrt{y} \cdot y - \left[\frac{1}{2}\left(\frac{1}{4}y\right)^2 + \frac{1}{4} y \cdot y \right] dy$$

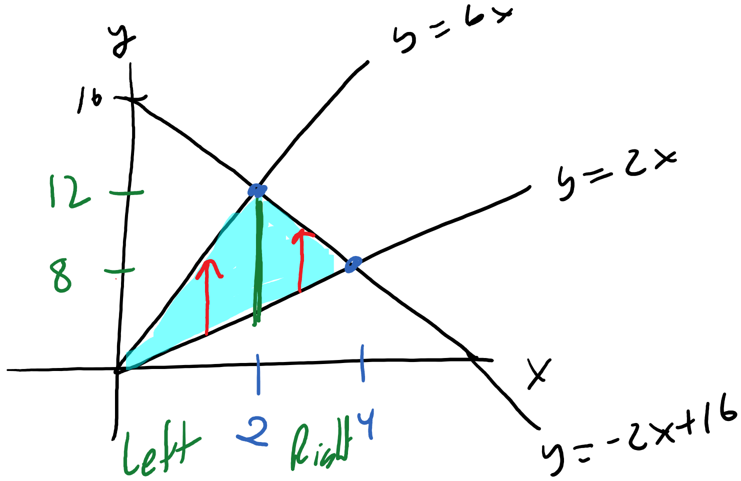
$$= \int_{y=0}^{16} \frac{y}{2} + y^{3/2} - \frac{1}{32} y^2 - \frac{1}{4} y^2 \, dy$$

$$= \int_{y=0}^{16} \frac{y}{2} + y^{3/2} - \frac{9}{32} y^2 \, dy$$

$$= \frac{y^2}{4} + \frac{2}{5} y^{5/2} - \frac{9}{32} \cdot \frac{1}{3} y^3 \Bigg|_{y=0}^{16}$$

$$= \frac{16^2}{4} + \frac{2}{5} (16)^{5/2} - \frac{3}{32} (16)^3 - (0) = 89.6$$

Example: Set up the double integral that would evaluate the function $f(x, y) = x + y$ over the region bounded by the lines $y = 6x$, $y = 2x$, and $y = -2x + 16$



$y = 6x$
$y = -2x + 16$

$6x = -2x + 16$
$8x = 16$
$x = 2$

$y = 2x$
$y = -2x + 16$

$2x = -2x + 16$
$4x = 16$
$x = 4$

$dy\,dx \uparrow$ $\qquad$ $dx\,dy \rightarrow$

Left

$0 \le x \le 2$

$2x \le y \le 6x$

Right

$2 \le x \le 4$

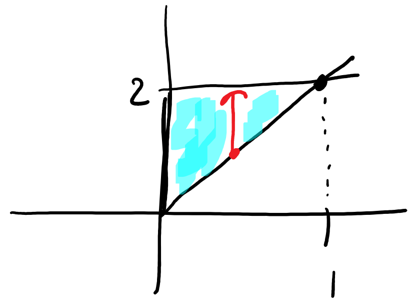
$2x \le y \le -2x + 16$

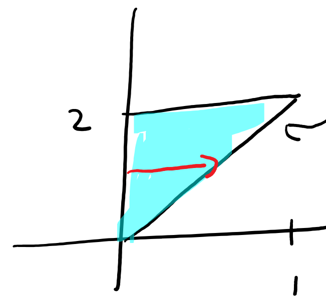
$$\iint_D x+y \, dA = \int_{x=0}^{2} \int_{y=2x}^{6x} x+y \, dy\, dx + \int_{x=2}^{4} \int_{y=2x}^{-2x+16} x+y \, dy\, dx$$

Example: Evaluate the integral by changing the order of integration $\int_0^1 \int_{2x}^2 \cos(y^2)\, dy dx$ $\longrightarrow$ $dx dy$

$0 \le x \le 1$

$\underbrace{2x \le}_{y=2x} \underbrace{y \le 2}_{y=2}$





$0 \le y \le 2$

$0 \le x \le \frac{1}{2} y$

$$\int_{x=0}^{1} \int_{y=2x}^{2} \cos(y^2)\, dy\, dx = \int_{y=0}^{2} \int_{x=0}^{\frac{1}{2}y} \cos(y^2)\, dx\, dy$$

$$= \int_{y=0}^{2} x \cos(y^2) \Big|_{x=0}^{\frac{1}{2}y} dy = \int_{y=0}^{2} \frac{1}{2} y \cos(y^2)\, dy$$

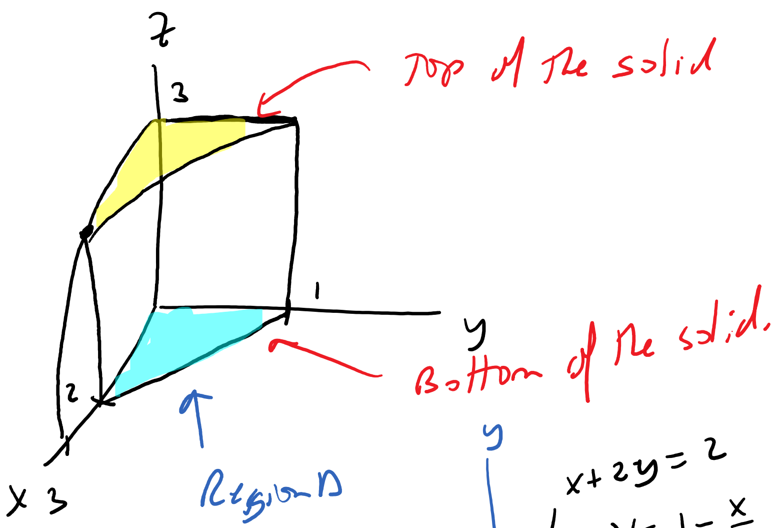
$u = y^2$

$$= \frac{1}{2} \cdot \frac{1}{2} \sin(y^2) \Big|_{y=0}^{2} = \frac{1}{4} \sin(4) - \frac{1}{4} \sin(0)$$

$$= \frac{1}{4} \sin(4)$$

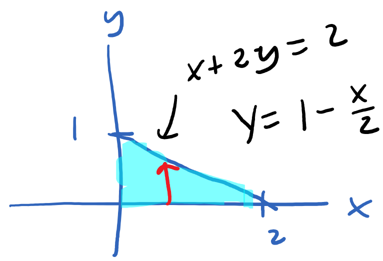
Example Find the volume of the solid bounded by the cylinder $x^2 + z^2 = 9$, the planes $x = 0$, $y = 0$, $z = 0$, $x + 2y = 2$ in the first octant.

$x=0$	yz-plane.
$y=0$	xz plane.
$z=0$	xy plane.



$$z^2 = 9 - x^2$$

$$z = \pm\sqrt{9 - x^2}$$



$$f(x,y) = z = \sqrt{9 - x^2}$$

↑ $dy\,dx$

$0 \le x \le 2$

$0 \le y \le 1 - \frac{x}{2}$

$$V = \iint_D \sqrt{9 - x^2}\, dA = \int_{x=0}^{2} \int_{y=0}^{1-\frac{x}{2}} \sqrt{9 - x^2}\, dy\, dx$$

$$= \int_{x=0}^{2} y\sqrt{9 - x^2}\,\Big|_{y=0}^{1-\frac{x}{2}} dx = \int_{x=0}^{2} \left(1 - \frac{x}{2}\right)\sqrt{9 - x^2}\, dx$$

$$= \int_{x=0}^{2} \sqrt{9 - x^2} - \frac{x}{2}\sqrt{9 - x^2}\, dx$$

$$= \int^{2} \qquad \int^{2} x \sqrt{\quad} \, dx$$

$$= \int_{x=0}^{2} \sqrt{9-x^2}\, dx - \int_{x=0}^{2} \frac{x}{2}\sqrt{9-x^2}\, dx$$

Trig sub.

$x = 3\sin\theta$

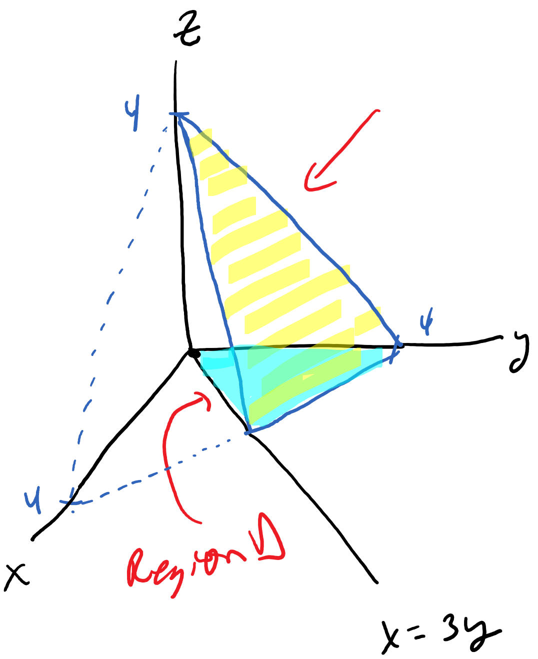
u-sub $u = 9-x^2$

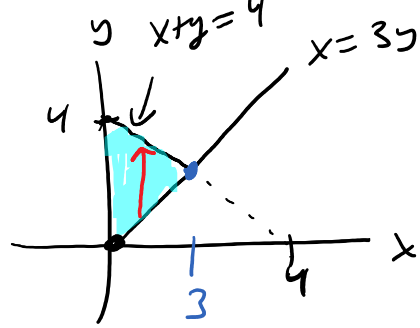
$\vdots$

$$= \frac{1}{6}\left(11\sqrt{5} - 27\right) + \frac{9}{2}\arcsin\left(\frac{2}{3}\right)$$

Example: Setup the integral(s) that would give the volumn of the solid (a tetrahedron) bounded by the planes $x = 0$, $z = 0$, $x = 3y$ and $x + y + z = 4$

Top of the solid $z = 4 - x - y$





$x+y=4 \to y=4-x$

$x=3y \to y=\frac{1}{3}x$

$dy\,dx$

$$0 \le x \le 3$$
$$\frac{1}{3}x \le y \le 4-x$$

$x+y=4$
$3y+y=4$
$4y=4$
$y=1$
$x=3$

$$\boxed{V = \int_{x=0}^{3} \int_{y=\frac{1}{3}x}^{4-x} 4-x-y \; dy\, dx}$$

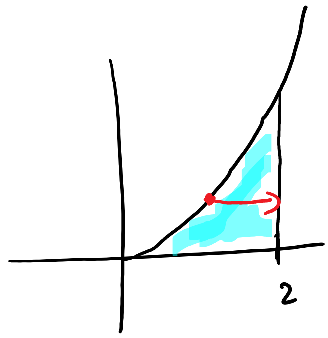
Example: Evaluate $\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy$

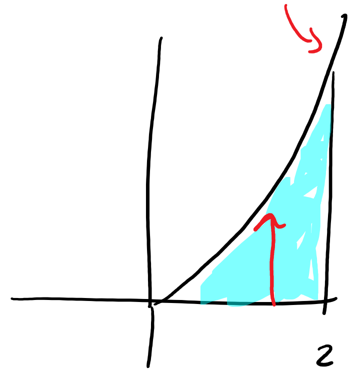
$x = y^{1/3} \to y = x^3$

$0 \le y \le 8 \to$

$\underbrace{\sqrt[3]{y}}_{x = y^{1/3}} \le x \le \underbrace{2}_{x=2}$

$y = x^3$





$0 \le x \le 2$

$0 \le y \le x^3$

$$\int_{y=0}^{8} \int_{x=\sqrt[3]{y}}^{2} e^{x^4}\, dx\, dy = \int_{x=0}^{2} \int_{y=0}^{x^3} e^{x^4}\, dy\, dx$$

$$= \int_{x=0}^{2} y e^{x^4} \Big|_{y=0}^{x^3} dx = \int_{x=0}^{2} x^3 e^{x^4}\, dx$$

$$= \frac{1}{4} e^{x^4} \Big|_0^2 = \frac{1}{4} e^{16} - \frac{1}{4} e^0$$

$$= \frac{1}{4} e^{16} - 1$$

Properties of Double Integrals:

- If a is a real number then $\displaystyle\iint_D (af(x,y) + g(x,y))dA = a\iint_D f(x,y)dA + \iint_D g(x,y)dA$
- If $D = D_1 \cup D_2$, where D_1 and D_2 do not overlap except perhaps on their boundaries then
$$\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$$
- $\displaystyle\iint_D 1\ dA = A(D) =$ the area of region D.