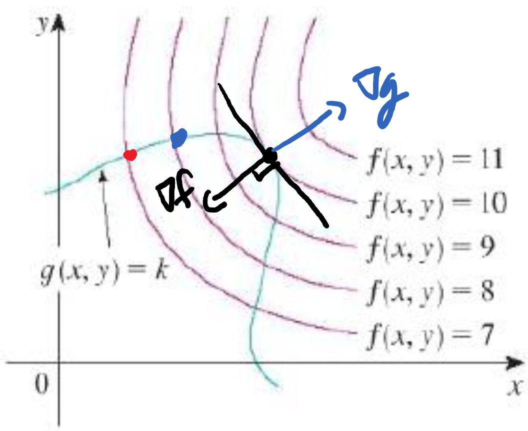
Section 14.8: Lagrange Multipliers

Lagrange multipliers are another method used to maximizing or minimizing a general function $f(x, y)$ subject to a constraint $g(x, y) = k$.

We are trying to find the extreme values of $f(x, y)$ subject to the condition $g(x, y) = k$. Thus we want to find the point(s) (a, b) on $g(x, y) = k$ so that the value of $f(a, b)$ will be a maximum or minimum.



$\nabla f = \lambda \nabla g$

Thus we want $f(x, y) = c$ some constant to just touch $g(x, y) = k$. This happens when both level curves have a common tangent line. This means that the normal lines where they touch, at the point (x_o, y_o) are identical. Thus

$\nabla f(x_o, y_o) = \lambda \nabla g(x_o, y_o)$ for some scalar λ.

The number λ is called a **Lagrange multiplier**.

Method of Lagrange multipliers: To find the maximum and minimum values of $f(x, y, z)$ subject to the constraints $g(x, y, z) = k$:

$\nabla g \neq 0$

(a) Find all values of x, y, z, and λ such that

$\nabla f = \lambda \nabla g$ and $g(x, y, z) = k$

$$f_x = \lambda g_x$$
$$f_y = \lambda g_y$$
$$f_z = \lambda g_z$$

(b) Evaluate f at all points (x, y, z) that arise from step (a). The largest of these values is the maximum value of f and the smallest is the minimum value of f.

Note: If there are two constraints, g and h, then $\nabla f = \lambda \nabla g + \mu \nabla h$

Example: Find the extreme values of the function $f(x,y) = x^2 + 2y^2$ on the ellipse $x^2 + 16y^2 = 16$

$\underbrace{x^2+16y^2}_{g(x,y)} = 16$

$f_x = \lambda g_x \qquad f_y = \lambda g_y$

$2x = \lambda \cdot 2x \qquad 4y = \lambda \cdot 32y \qquad x^2 + 16y^2 = 16$

$2x - 2x\lambda = 0$

$2x(1-\lambda) = 0$

$x = 0 \qquad \lambda = 1$

$0 + 16y^2 = 16$

$y^2 = 1$

$y = \pm 1$

$(0,1)$

$(0,-1)$

$4y = 32y$

$0 = 28y$

$y = 0$

$x^2 = 16$

$x = \pm 4$

$(4,0)$

$(-4,0)$

$4y - 32\lambda y = 0$

$4y(1 - 8\lambda) = 0$

$y = 0 \qquad \lambda = \frac{1}{8}$

$2x = \frac{1}{8} \cdot 2x$

$2x = \frac{1}{4}x$

$x = 0$

points	$f(x,y) = x^2 + 2y^2$
$(0,1)$	2
$(0,-1)$	2
$(4,0)$	16
$(-4,0)$	16

min value (for $(0,1)$, $(0,-1)$)

max value (for $(4,0)$, $(-4,0)$)

Example: Find the points on the sphere $x^2 + y^2 + z^2 = 25$ where $f(x, y, z) = x + 2y + 3z$ has its maximum and minimum value.

$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z$

$g(x,y,z) = 25$

$x^2+y^2+z^2=25$

$1 = \lambda 2x \qquad 2 = \lambda 2y \qquad 3 = \lambda 2z$

$x, y, z, \lambda \neq 0$

$\lambda = \frac{1}{2x} \qquad \lambda = \frac{1}{y} \qquad \lambda = \frac{3}{2z}$

$\frac{1}{2x} = \frac{1}{y}$

$y = 2x$

$\frac{1}{2x} = \frac{3}{2z}$

$2z = 6x$

$z = 3x$

$x^2 + (2x)^2 + (3x)^2 = 25$

$x^2 + 4x^2 + 9x^2 = 25$

$14x^2 = 25$

$x^2 = \frac{25}{14}$

$x = \pm \frac{5}{\sqrt{14}}$

	points	
$x = \frac{5}{\sqrt{14}}$	$\left(\frac{5}{\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{15}{\sqrt{14}}\right)$	$f(x,y,z) > 0$ max.
$x = -\frac{5}{\sqrt{14}}$	$\left(\frac{-5}{\sqrt{14}}, \frac{-10}{\sqrt{14}}, \frac{-15}{\sqrt{14}}\right)$	$f(x,y,z) < 0$ min.

$x = \frac{1}{2\lambda} \qquad y = \frac{1}{\lambda} \qquad z = \frac{3}{2\lambda}$

$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = 25$

$$\left(\frac{1}{2x}\right)^2 + \left(\frac{1}{x}\right)^2 + \left(\frac{3}{2x}\right)^2 = 25$$

Algebra is done $\quad x = \pm \frac{\sqrt{7}}{\sqrt{50}}$

$x = \frac{5}{\sqrt{14}}$ . . . . - -

$x = -\frac{5}{\sqrt{14}}$ . . . - - -

Example: Find the point on the sphere $x^2 + y^2 + z^2 = 9$ that is closest to the point $(2, 3, 4)$.

$$d = \sqrt{(x-2)^2 + (y-3)^2 + (z-4)^2}$$

$$f(x,y,z) = d^2 = (x-2)^2 + (y-3)^2 + (z-4)^2$$

$$\underbrace{x^2+y^2+z^2}_{g(x,y,z)} = 9$$

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z$$

$$2(x-2) = \lambda 2x \qquad 2(y-3) = \lambda 2y \qquad 2(z-4) = \lambda 2z$$

$$2y(x-2) = 2\lambda xy \qquad 2x(y-3) = 2\lambda xy$$

$$2y(x-2) = 2x(y-3)$$

$$xy - 2y = xy - 3x$$

$$-2y = -3x$$

$$y = \frac{3}{2}x$$

$$2z(x-2) = 2\lambda xz = 2x(z-4)$$

$$2z(x-2) = 2x(z-4)$$

$$zx - 2z = zx - 4x$$

$$-2z = -4x$$

$$z = 2x$$

$$x^2 + y^2 + z^2 = 9$$

$$x^2 + \left(\frac{3}{2}x\right)^2 + (2x)^2 = 9$$

$$4\left(x^2 + \frac{9}{4}x^2 + 4x^2 = 9\right)$$

$$4x^2 + 9x^2 + 16x^2 = 36$$

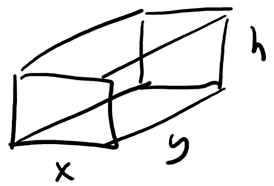
$$29x^2 = 36$$

$$x^2 = \frac{36}{29}$$

$$x = \pm \frac{6}{\sqrt{29}}$$

$x = \frac{6}{\sqrt{29}}$	$x = -\frac{6}{\sqrt{29}}$
$y = \frac{9}{\sqrt{29}}$	$y = \frac{-9}{\sqrt{29}}$
$z = \frac{12}{\sqrt{29}}$	$z = \frac{-12}{\sqrt{29}}$
min	max

Example: The base of a rectangular tank with volume of 540 cubic units is made of slate and the sides are made of glass. If slate costs five times as much as glass(per unit area), find the dimensions of the tank which minimize the cost of the materials.



$$C = 5xy + 2xh + 2yh$$

$$V = \underbrace{xyh}_{g(x,y,z) = xyh} = 540$$

$$C_x = \lambda g_x \qquad C_y = \lambda g_y \qquad C_h = \lambda g_h$$

$$5y + 2h = \lambda yh \qquad 5x + 2h = \lambda xh \qquad 2x + 2y = \lambda xy$$

$$x(5y + 2h) = \lambda xyh = y(5x + 2h)$$

$$5xy + 2hx = 5xy + 2yh$$

$$2hx = 2yh$$

$$h = 0 \quad \text{or} \quad 2x = 2y$$

$$x = y$$

$$x(5y + 2h) = \lambda xyh = h(2x + 2y)$$

$$5xy + 2xh = 2xh + 2yh$$

$$5xy = 2yh$$

$$y = 0 \quad \text{or} \quad 5x = 2h$$

$$h = \frac{5x}{2}$$

$$V = xyh = 540$$

$$x \cdot x \cdot \frac{5x}{2} = 540$$

$$x^3 = \frac{1080}{5} = 216$$

$$x = \sqrt[3]{216} = 6$$

$$y = 6$$

$$h = \frac{5(6)}{2} = 15$$

$$\boxed{6, 6, 15}$$