Chain rule for functions of a single variable: If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Example: Let $z = x^y$ with $x = t^3$ and $y = \sin(t)$. Compute $z'(t) = \frac{dz}{dt}$.

$$z = (t^3)^{\sin(t)} = t^{3\sin(t)}$$

$$\ln z = \ln t^{3\sin t}$$

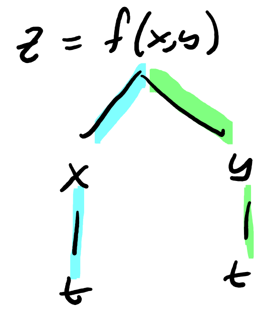
$$\ln(z) = 3\sin t \cdot \ln(t)$$

$$\frac{z'}{z} = 3\cos(t)\ln(t) + \frac{3\sin(t)}{t}$$

$$z' = \frac{dz}{dt} = z\left[3\cos(t)\ln(t) + \frac{3\sin(t)}{t}\right]$$

$$\frac{dz}{dt} = t^{3\sin(t)}\left[3\cos(t)\cdot\ln(t) + \frac{3\sin(t)}{t}\right]$$

Chain Rule (Case 1): Suppose the $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and



$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \qquad \text{or} \qquad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Example: Let $z = x^y$ with $x = t^3$ and $y = \sin(t)$. Compute $z'(t) = \dfrac{dz}{dt}$

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}$$

$$= y \cdot x^{y-1}(3t^2) + x^y \cdot 1 \ln(x) \cdot (\cos(t))$$

$$\frac{dz}{dt} = y\, x^{y-1} \cdot 3t^2 + x^y \ln(x) \cdot \cos(t)$$

Example: Let $z = \ln(x + y^2)$ with $x = \sqrt{1+t^2}$ and $y = e^{3t}$. Compute $z'(t) = \frac{dz}{dt}$

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}$$

$$= \frac{1}{x+y^2} \cdot \frac{1}{2}(1+t^2)^{-1/2} \cdot 2t + \frac{2y}{x+y^2} \cdot 3e^{3t}$$

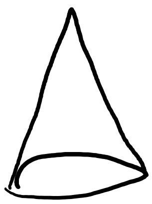
$$= \frac{t}{(x+y^2)\sqrt{1+t^2}} + \frac{6ye^{3t}}{x+y^2}$$

Example: The radius of a right circular cone is increasing at a rate of 2.1 in/s while its height is decreasing at a rate of 1.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in and the height is 140 in?

$\frac{dr}{dt} = 2.1$ in/sec

$\frac{dh}{dt} = -1.5$ in/sec

$r = 120 \quad h = 140$



$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = V_r \frac{dr}{dt} + V_h \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{2}{3}\pi r h \cdot \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$$

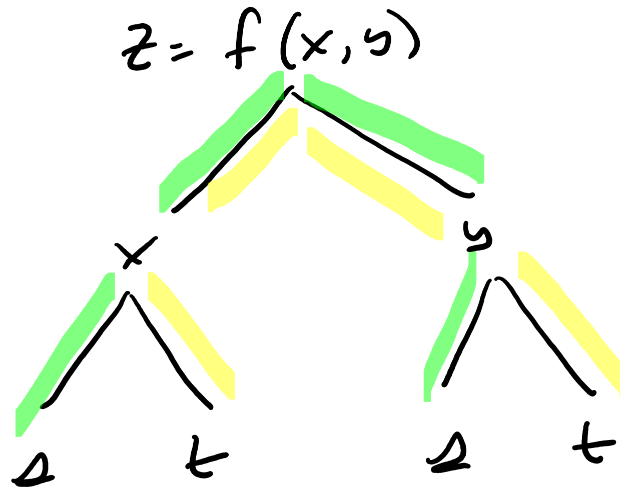
$$= \frac{2}{3}\pi(120)(140)(2.1) + \frac{1}{3}\pi(120)^2(-1.5)$$

$$\frac{dV}{dt} = 163.20\pi \text{ cubic in/sec}$$

Chain Rule (Case 2): Suppose the $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ and all first partials of x and y exists. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = z_x x_s + z_y y_s$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



Example: $z = \sin(x)\cos(y)$, where $x = (s-t)^2$ and $y = s^2 - t^2$. Compute z_s and z_t.

$z_x = \cos(x)\cos(y)$

$z_y = -\sin(x)\sin(y)$

$x_s = 2(s-t)^1$

$x_t = 2(s-t)^1(-1)$

$y_s = 2s$

$y_t = -2t$

$z_s = z_x x_s + z_y y_s$

$z_s = \cos(x)\cos(y)\cdot 2(s-t) - \sin(x)\sin(y)\ (2s)$

$z_t = z_x x_t + z_y y_t$

$z_t = \cos(x)\cos(y)\cdot(-2)(s-t) + \sin(x)\sin(y)\ (2t)$

$x=2 \quad y=4 \quad z=0$

Example: If $u = x^2y + y^3z^2$ where $x = rse^t$, $y = rt^3 + s^2$ and $z = rs\sin(t)$, find u_s when $(r, s, t) = (1, 2, 0)$.

$r=1 \quad t=0$
$s=2$

$u_s = u_x x_s + u_y y_s + u_z z_s$

$= (16)(1) + 4(4) + (0)(0)$

$= 16 + 16$

$= 32$

$u_x = 2xy \longrightarrow u_x = 16$

$u_y = x^2 + 3y^2z^2 \longrightarrow u_y = 4$

$u_z = 2y^3z \longrightarrow u_z = 0$

$x_s = re^t \longrightarrow x_s = 1$

$y_s = 2s \longrightarrow y_s = 4$

$z_s = r\sin(t) \longrightarrow z_s = 0$

Example: Suppose f is a differentiable function of x and y, and $g(a,b) = f(e^a + \sin(b), e^a + \cos(b))$. Use the table of values to calculate $g_a(0,0)$ and $g_b(0,0)$.

	f	g	f_x	f_y
$(0,0)$	4	5	10	20
$(1,2)$	8	9	7	6

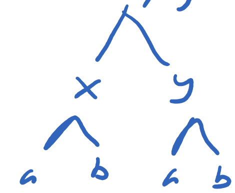
$x = e^a + \sin(b)$

$y = e^a + \cos(b)$

$x = 1$ $\quad a = 0$

$y = 2$ $\quad b = 0$

$g(a,b) = f(x,y)$



$a = 0$, $b = 0$

$x_a = e^a \longrightarrow x_a = 1$

$x_b = \cos(b) \longrightarrow x_b = 1$

$y_a = e^a \longrightarrow y_a = 1$

$y_b = -\sin(b) \longrightarrow y_b = 0$

$g_a = f_x\, x_a + f_y\, y_a$

$g_a(\overset{a}{0},\overset{b}{0}) = f_x(1,2)\, x_a + f_y(1,2)\, y_a$

$= 7(1) + 6(1) = 13$

$g_b(0,0) = f_x(1,2)\, x_b + f_y(1,2)\, y_b$

$= 7(1) + (6)(0) = 7$

Implicit Differentiation: Suppose that an equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x. i.e. $y = f(x)$ and $F(x, f(x)) = 0$ for all x in the domain of $f(x)$. Find $\frac{dy}{dx}$.

$$F_x \underbrace{\frac{dx}{dx}} + F_y \frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = \frac{-F_x}{F_y}}$$

$$F_x + F_y \underbrace{\frac{dy}{dx}} = 0$$

Example: Find $\dfrac{dy}{dx}$ if $x^3 + y^3 = 6x^2y^4$

$$\underbrace{x^3 + y^3 - 6x^2y^4}_{F(x,y)} = 0$$

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(3x^2 - 12xy^4)}{3y^2 - 24x^2y^3}$$

Example: Suppose that z is given implicitly as a function $z = f(x, y)$ by an equation $F(x, y, z) = 0$. Find z_x and z_y.

$\frac{\partial z}{\partial x}$

$$F_x \underbrace{\frac{\partial x}{\partial x}}_{1} + F_y \underbrace{\frac{\partial y}{\partial x}}_{0} + F_z \frac{\partial z}{\partial x} = 0$$

$$F_x + F_z \frac{\partial z}{\partial x} = 0$$

$$z_x = \frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$$

$$z_y = \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

Example: If $\overbrace{x^4 + y^3 + z^2 + xye^z}^{F(x,y,z)} = 10$. Find

(a) $z_x = -\dfrac{F_x}{F_z} = \dfrac{-(4x^3 + ye^z)}{2z + xye^z}$ $\quad z(x,y)$

(b) $x_y = \dfrac{-F_y}{F_x} = \dfrac{-(3y^2 + xe^z)}{4x^3 + ye^z}$ $\quad x(y,z)$