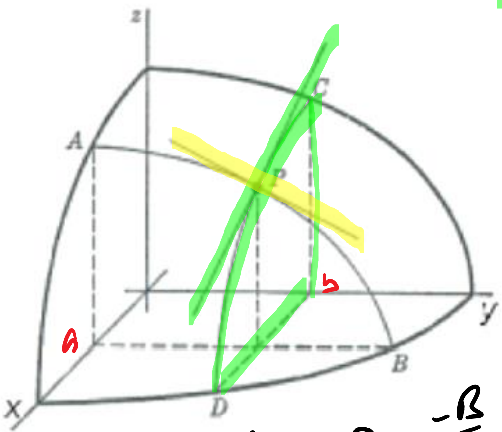
Definition: Suppose the surface S has the equation $z = f(x, y)$, where f has continuous first partials, and let $P(a, b, c)$ be a point on the surface. If C_1 and C_2 be the curves obtained by intersecting the planes $x = a$ and $y = b$ with the surface, then T_1 and T_2 are the respective tangent lines to the curves at point P. The **tangent plane** to the surface S at the point $P(a, b, c)$ is defined to be the plane that contains both of the tangent lines at point P.

f_x $\quad V_{f_x} = \langle 1, 0, f_x(a,b) \rangle$

f_y $\quad V_{f_y} = \langle 0, 1, f_y(a,b) \rangle$



$A(x-a) + B(y-b) + C(z-c) = 0$

$\boxed{z - c = A_1(x-a) + B_1(y-b)}$

$z = A_1(x-a) + B_1(y-b) + c$

$A_1 = \frac{-A}{C} \qquad B_1 = \frac{-B}{C}$

$\frac{\partial z}{\partial x} = A_1 \qquad \frac{\partial z}{\partial y} = B_1$

$\hookrightarrow \frac{\partial z}{\partial x} = f_x(a,b) = \underline{\underline{A_1}} \qquad \frac{\partial z}{\partial y} = f_y(a,b) = \underline{\underline{B_1}}$

$n = \langle A_1, B_1, -1 \rangle = \langle f_x(a,b), f_y(a,b), -1 \rangle$

Theorem: An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(a, b, c)$ or $P(a, b, f(a, b))$ is

$f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z-c) = 0$

The normal vector for this plane is

$n = \langle f_x, f_y, -1 \rangle$ at point (a, b)

The **normal line** is the line though the point $P(a, b, c)$ that is perpendicular to the tangent plane.

$$x = a + f_x(a,b) \cdot t$$

$$y = b + f_y(a,b) \cdot t$$

$$z = c - t$$

Example: Find an equation of the tangent plane to the graph of the function $z = 3x^2 + y^4$ at the point $(2, 1, 13)$.

$z_x = 6x$ $\quad z_x(2,1) = 12$ $\quad n = \langle 12, 4, -1 \rangle$

$z_y = 4y^3$ $\quad z_y(2,1) = 4$

$$12(x-2) + 4(y-1) - (z-13) = 0$$

or $\quad 12x + 4y - z = 15$

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normal line

$x = 2 + 12t$
$y = 1 + 4t$
$z = 13 - t$

Example: Find an equation of the tangent plane $f(x, y) = \ln(5x + 2y)$ at the point $(-1, 3, 0)$. Also find a formula for the normal line at this point.

$$f_x = \frac{5}{5x+2y} \qquad f_x(-1,3) = \frac{5}{-5+6} = \frac{5}{1} = 5$$

$$f_y = \frac{2}{5x+2y} \qquad f_y(-1,3) = \frac{2}{1} = 2 \qquad n = \langle 5, 2, -1 \rangle$$

tangent plane $\quad 5x + 2y - z = 1$

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normal line

$$x = -1 + 5t$$
$$y = 3 + 2t$$
$$z = -t$$

$\frac{dy}{dx} = f'(x)$

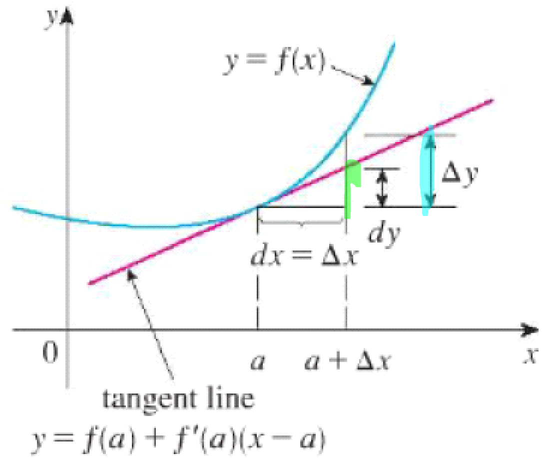
Differentials:

In Cal I we had for $y = f(x)$ the differentials dy and dx defined as

$$dy = f'(x)dx$$

We saw that for the point (a, f(a)) and $dx = \Delta x$ we found that

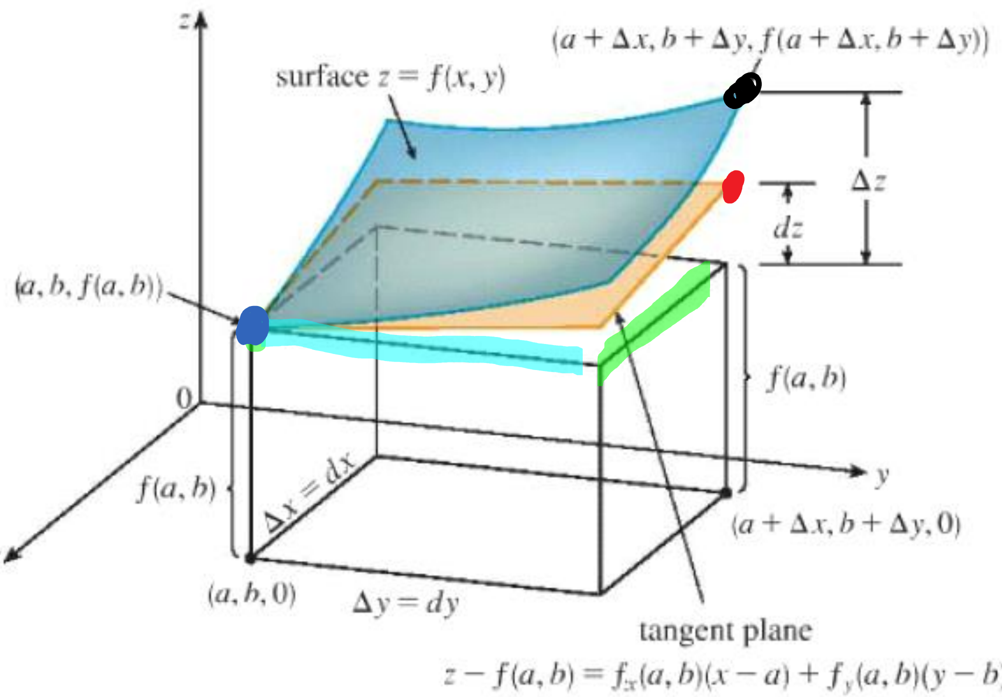
$$f(a + \Delta x) = f(a) + \Delta y \approx f(a) + dy = L(x)$$



Definition: Consider a function of two variables $z = f(x, y)$. Let Δx and Δy be increments of x and y, respectively.

- Then **differentials dx and dy** are independent variables and $dx = \Delta x$ and $dy = \Delta y$
- The **differential dz**, also called the **total differential**, is a dependent variable and is defined by $\quad dz = f_x(x, y)dx + f_y(x, y)dy$

Note: $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$ and when Δx and Δy are small and the partials are both continuous, then $\Delta z \approx dz$.



The linearization, $L(x, y)$, of the function $f(x, y)$ at the point (a, b) is

$$L(x, y) \approx f(a, b) + \underbrace{f_x(a, b)\underbrace{(x - a)}_{dx} + f_y(a, b)\underbrace{(y - b)}_{dy}}_{dz}$$

Example: Use differentials to find an approximate value for $\sqrt{1.03^2 + 1.98^3}$

$f(x,y) = \sqrt{x^2+y^3}$

Start point $(1,2)$ end point $(1.03, 1.98)$

$\Delta x = dx = 1.03 - 1 = .03$

$\Delta y = dy = 1.98 - 2 = -.02$

$f(1,2) = \sqrt{1^2 + 2^3}$

$= \sqrt{1+8}$

$= \sqrt{9} = 3$

$f_x = \frac{1}{2}(x^2+y^3)^{-1/2}(2x)$

$f_y = \frac{1}{2}(x^2+y^3)^{-1/2}(3y^2)$

$f_x = \dfrac{x}{\sqrt{x^2+y^3}}$

$f_y = \dfrac{3y^2}{2\sqrt{x^2+y^3}}$

$f_x(1,2) = \dfrac{1}{3}$

$f_y = \dfrac{3(2)^2}{2(3)} = \dfrac{12}{6} = 2$

$dz = f_x\,dx + f_y\,dy$

$= \frac{1}{3}(.03) + 2(-.02)$

$= .01 - .04 = -.03$

$f(1.03, 1.98) \approx f(1,2) + dz = 3 + (-.03)$

$= 2.97$

$$\Delta z = f(1.03, 1.58) - f(1.2) = -.029597$$

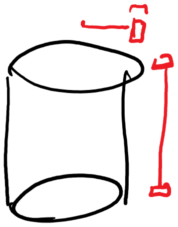
$$L(x,y) = f(1,2) + \frac{1}{3}(x-1) + 2(y-2)$$

$$= 3 + \frac{1}{3}(x-1) + 2(y-2)$$

Example: Find the differential(i.e. the total differential) of the function $w = x^5y^3 + x^2z^4$.

$$dw = w_x\,dx + w_y\,dy + w_z\,dz$$
$$= (5x^4y^3 + 2xz^4)\,dx + 3x^5y^2\,dy + 4x^2z^3\,dz$$

Example: Estimate the amount of material in a closed can (right circular cylinder) with a radius of 3 in and a height of 8 in if the material of the can is 0.04 in thick.



$V = \pi r^2 h$

$r = 3$ $\quad$ $dr = .04$

$h = 8$ $\quad$ $dh = 2(.04)$
$= .08$

$dV = \pi \cdot 2rh\,dr + \pi r^2\,dh$

$= \pi\, 2(3)(.04) + \pi(3)^2(.08)$

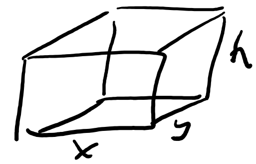
$= 2.64\pi$

8.29 cubic inches.

If $dr = -.04$ $\quad$ $dh = -.08$ then

$dV = -8.29$ cubic in $\longrightarrow$ 8.29 cubic in

Example: The dimensions of a closed rectangular box are measured as 90cm, 70cm, and 60cm with a possible error of 0.3cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.



$SA = 2xy + 2yh + 2xh$

$dSA = (2y + 2h)\,dx + (2x + 2h)\,dy + (2y + 2x)\,dh$

$x = 90$

$y = 70$

$h = 60$

$dx = dy = dh = .3$

$dSA = 264$ sq cm.

Definition: A function $f(x, y)$ is **differentiable** at (a, b) if its partial derivatives f_x and f_y exist and are continuous at (a, b).

Note: Polynomial and rational functions are differentiable on their domains.

Example: Find an equation of a tangent plane for the surface at the point $(1, 1, 1)$, if it is know the two space curves $\mathbf{r}(t)$ and $\mathbf{g}(s)$ are both both on the surface and they both go though the point $(1, 1, 1)$

$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ $\quad t = 1$

$1 + 2s = 1$

$2s = 0$

$s = 0$

$\mathbf{g}(s) = \langle 1 + 2s, 1 + s - s^2, 1 - s + s^2 - s^3 \rangle$ $\quad s = 0$

$r' = \langle 1, 2t, 3t^2 \rangle$ $\qquad g'(s) = \langle 2, 1 - 2s, -1 + 2s - 3s^2 \rangle$

$r'(1) = \langle 1, 2, 3 \rangle$ $\qquad g'(0) = \langle 2, 1, -1 \rangle$

$$n = r'(1) \times g'(0) = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & 1 & -1 \end{vmatrix} = \cdots = \langle -5, 7, -3 \rangle$$

$$-5(x-1) + 7(y-1) - 3(z-1) = 0$$