Section 13.3: Arc Length and Curvature

In Cal II, the arc length of a two-dimensional smooth curve that is only traversed once on an interval $I$ was given by

$$L = \int_I ds \text{ or } L = \int_I \sqrt{[x'(t)]^2 + [y'(t)]^2}\ dt$$

x(t)
y(t)

This can be extended to a space curve. If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ on the interval $a \le t \le b$, then the length of the curve is given by

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\ dt \qquad \text{or} \qquad \boxed{L = \int_a^b |\mathbf{r}'(t)|\ dt.}$$

Note: A curve $\mathbf{r}(t)$ is called smooth on an interval if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on the interval. A smooth curve has no sharp corners or cusps, i.e. the tangent vector has continuous movement. The arc length formula holds for smooth and piecewise-smooth curves.

Example: Find the length of the arc for $\mathbf{r}(t) = \langle 3t, 2\sin(t), 2\cos(t)\rangle$ from the point $(0,0,2)$ to $(6\pi, 0, 2)$.

$t=0$ $\quad$ $t=2\pi$

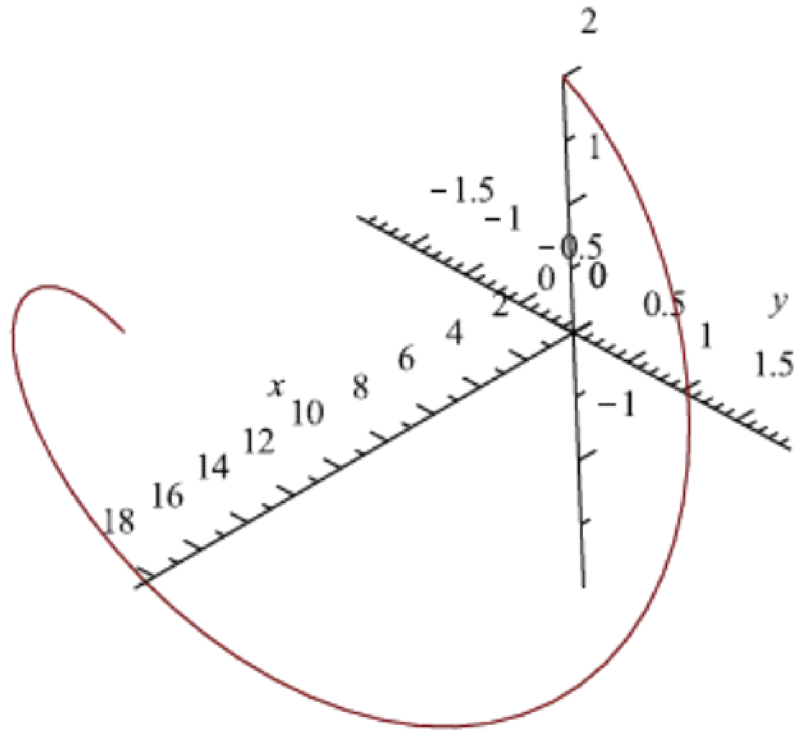
$r' = \langle 3, 2\cos(t), -2\sin(t)\rangle$

$$L = \int_0^{2\pi} |r'(t)|\,dt = \int_0^{2\pi} \sqrt{9 + \underbrace{4\cos^2(t) + 4\sin^2(t)}}\;dt$$

$$= \int_0^{2\pi} \sqrt{9+4}\;dt = \int_0^{2\pi} \sqrt{13}\;dt = t\sqrt{13}\Big|_0^{2\pi}$$

$$L = 2\pi\sqrt{13}$$

graph



**Definition:** The **arc length function**, $s$, is $s(t) = \int_a^t |\mathbf{r}'(u)|\,du$. where $t=a$ is the start.

The arc length $s$ is called the **arc length parameter.**

Example: Find the arc length function for $\mathbf{r}(t) = \langle e^t, e^t\sin(t), e^t\cos(t)\rangle$ from the point $(1,0,1)$ in the direction of increasing $t$.

$\hookrightarrow t=0$

$r'(t) = \langle e^t, e^t\sin(t) + e^t\cos t, e^t\cos t - e^t\sin t\rangle$

$$s = \int_0^t |r'(u)|\,du$$

$$= \int_0^t \sqrt{(e^u)^2 + (e^u\sin(u) + e^u\cos u)^2 + (e^u\cos u - e^u\sin(u))^2}\,du$$

$$= \int_0^t \sqrt{e^{2u} + e^{2u}\sin^2 u + 2e^{2u}\cos u\sin u + e^{2u}\cos^2 u + e^{2u}\cos^2 u - 2e^{2u}\cos u\sin u + e^{2u}\sin^2 u}\,du$$

$$= \int_0^t \sqrt{e^{2u} + 2e^{2u}\sin^2 u + 2e^{2u}\cos^2 u}\,du$$

$$= \int_0^t \sqrt{e^{2u} + 2e^{2u}}\,du = \int_0^t \sqrt{3e^{2u}}\,du$$

$$s = \int_0^t \sqrt{3}e^u\,du = \sqrt{3}e^u\Big|_0^t$$

$$y = \sqrt{3}\, e^{t} - \sqrt{3}$$

Example: Reparametrize the curve $\mathbf{r}(t) = \langle 1+2t,\ 3+t,\ -5t\rangle$ with respect to arc length measured from the point where $t=0$ in the direction of increasing $t$.

$$r' = \langle 2, 1, -5\rangle$$

$$s = \int_0^t |r'(u)|\,du = \int_0^t \sqrt{4+1+25}\,du$$

$$= \int_0^t \sqrt{30}\,du = u\sqrt{30}\Big|_0^t = t\sqrt{30}$$

$$s = t\sqrt{30}$$

$$t = \frac{s}{\sqrt{30}}$$

$(1, 3, 0)$

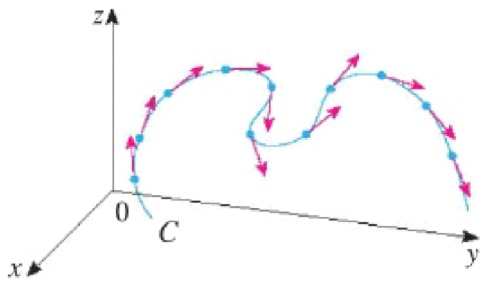
$$r(t) = \langle 1+2t,\ 3+t,\ -5t\rangle$$

$$r(s) = \left\langle 1+\frac{2s}{\sqrt{30}},\ 3+\frac{s}{\sqrt{30}},\ \frac{-5s}{\sqrt{30}}\right\rangle$$

Pg 5: curvature

**Definition:** The **curvature**, $\kappa$, of a curve is defined to be the magnitude of the rate of change of the unit tangent vector with respect to the arc length is given by $\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$

$$\left|\frac{dT}{ds}\right| = \left|\frac{\frac{dT}{dt}}{\frac{ds}{dt}}\right| = \frac{|T'(t)|}{|r'(t)|}$$




Unit tangent vectors at equally spaced points on $C$

$$s(t) = \int_0^t |r'(u)|\,du \to \frac{ds}{dt} = |r'(t)| \cdot 1$$

**Theorem** The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = |\mathbf{r}''(s)|$$

1 2 3

Example: Find the curvature of $\mathbf{r}(t) = \left\langle -\sqrt{2}\sin t,\ \cos t,\ \cos t \right\rangle$.

$$r'(t) = \langle -\sqrt{2}\cos t, -\sin t, -\sin t \rangle$$

$$|r'(t)| = \sqrt{2\cos^2 t + \sin^2 t + \sin^2 t} = \sqrt{2\cos^2 t + 2\sin^2 t} = \sqrt{2}$$

$$T = \frac{1}{\sqrt{2}} \langle -\sqrt{2}\cos t, -\sin t, -\sin t \rangle$$

$$T' = \frac{1}{\sqrt{2}} \langle \sqrt{2}\sin t, -\cos t, -\cos t \rangle$$

$$= \left\langle \sin t, -\frac{\cos t}{\sqrt{2}}, -\frac{\cos t}{\sqrt{2}} \right\rangle$$

$$|T'| = \sqrt{\sin^2 t + \frac{\cos^2(t)}{2} + \frac{\cos^2(t)}{2}}$$

$$= \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$K = \frac{|T'|}{|r'|} = \frac{1}{\sqrt{2}}$$

Example: Find the curvature of $\mathbf{r}(t) = \left\langle 1+t, 1-t, 3t^2 \right\rangle$

$$r'' = \langle 0, 0, 6 \rangle$$

$$r' = \langle 1, -1, 6t \rangle$$

$$|r'| = \sqrt{1 + 1 + 36t^2} = \sqrt{2 + 36t^2}$$

$$T = \frac{1}{\sqrt{2+36t^2}} \langle 1, -1, 6t \rangle = \left\langle \frac{1}{\sqrt{2+36t^2}}, \frac{-1}{\sqrt{2+36t^2}}, \frac{6t}{\sqrt{2+36t^2}} \right\rangle$$

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$$r' \times r'' = \begin{vmatrix} i & j & k \\ 1 & -1 & 6t \\ 0 & 0 & 6 \end{vmatrix} = \cdots = \langle -6, -6, 0 \rangle$$

$$K = \frac{|r' \times r''|}{|r'|^3} = \frac{\sqrt{36+36+0}}{\left(\sqrt{2+36t^2}\right)^3} = \frac{\sqrt{72}}{(2+36t^2)^{3/2}}$$

Note: The **unit normal vector** is defined as $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$ and the **binormal vector** is defined as $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$. $\mathbf{B}(t)$ is also a unit vector.

The plane determined by $\mathbf{N}(t)$ and $\mathbf{B}(t)$ is called the **normal plane**.
The plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$ is called the **osculating plane**.