Section 12.3: The Dot Product

Definition: The **dot product** of two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ is the number

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, $0 \le \theta \le \pi$. If either $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$, then $\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$.

The **dot product** of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Definition: Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal(perpendicular) if and only if $\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$, i.e. the angle between them is $\pi/2$.

Example: Find the following using these vectors: $\mathbf{a} = \langle -1, -2, -3 \rangle$, $\mathbf{b} = \langle -10, 2, 1 \rangle$, and $\mathbf{c} = \langle 2, 8, -6 \rangle$.

A) $\mathbf{a} \cdot \mathbf{b} =$ $(-1)(-10) + (-2)(2) + (-3)(1) = 10 - 4 - 3 = 3$

B) $\mathbf{a} \cdot \mathbf{c} =$ $(-1)(2) + (-2)(8) + (-3)(-6) = -2 - 16 + 18 = 0$

a & c are perp.

C) Find the angle between **a** and **b**.

$a \cdot b = |a|\,|b| \cos\theta$

$|a| = \sqrt{1 + 4 + 9} = \sqrt{14}$

$|b| = \sqrt{100 + 4 + 1} = \sqrt{105}$

$3 = \sqrt{14}\sqrt{105} \cos\theta$

$\cos\theta = \dfrac{3}{\sqrt{14}\sqrt{105}}$

$\theta = \arccos\left(\dfrac{3}{\sqrt{14}\sqrt{105}}\right)$

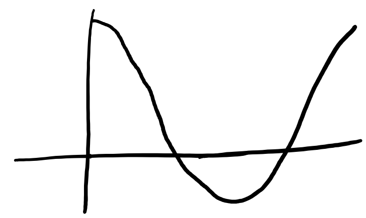
$\theta = 1.4925$ rad.

85.57 degrees.

Example: If $|\mathbf{a}| = 1$ and $|\mathbf{b}| = 2$, what is the maximum for $\mathbf{a} \cdot \mathbf{b}$? What does this say about the vectors?

$$a \cdot b = |a|\,|b| \cos\theta$$
$$= 2\cos\theta$$

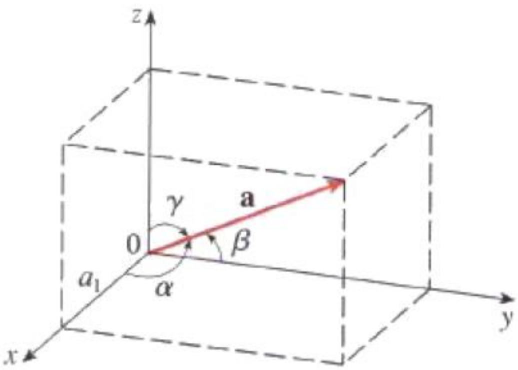
$$\cos\theta = 1$$
$$\theta = 0^\circ$$



max $= 2$

Both vectors point in the same direction.

Directional angles/and Direction Cosines



$a = \langle a_1, a_2, a_2 \rangle$

$i = \langle 1, 0, 0 \rangle$

$a \cdot i = |a||i| \cos \alpha$

$a_1 = |a| \cos \alpha$

$\cos \alpha = \dfrac{a_1}{|a|}$

$\beta = \cos^{-1}\left(\dfrac{a_2}{|a|}\right) \quad \gamma = \cos^{-1}\left(\dfrac{a_3}{|a|}\right) \quad \alpha = \cos^{-1}\left(\dfrac{a_1}{|a|}\right)$

Example: Find the direction angles for $\mathbf{a} = \langle 1, 0, 5 \rangle$

$\alpha = \cos^{-1}\left(\dfrac{1}{\sqrt{26}}\right)$

$|a| = \sqrt{1+0+25}$
$= \sqrt{26}$

$\beta = \cos^{-1}\left(\dfrac{0}{\sqrt{26}}\right) = \cos^{-1}(0) = \dfrac{\pi}{2}$

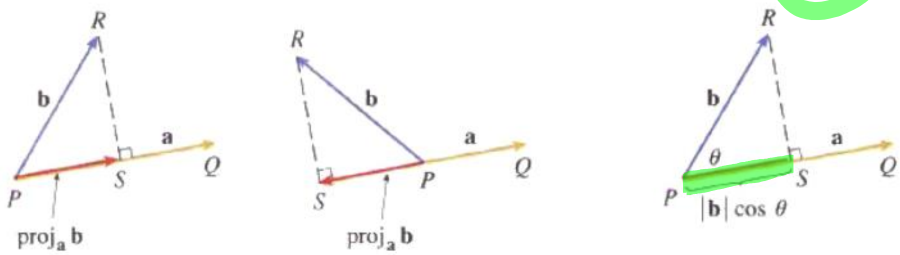
$\gamma = \cos^{-1}\left(\dfrac{5}{\sqrt{26}}\right)$

Projections

Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$: $\quad \text{comp}_{\mathbf{a}}\, \mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

Vector projection of $\mathbf{b}$ onto $\mathbf{a}$: $\quad \text{proj}_{\mathbf{a}}\, \mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$

$$\left(\frac{a \cdot b}{|a|}\right) \frac{a}{|a|}$$



Vector projections

Scalar projection

Example: Find the vector and scalar projections of $\mathbf{m} = \langle 2, 1, 5 \rangle$ onto $\mathbf{n} = \langle 1, 2, 3 \rangle$

$$\text{comp}_n m = \frac{m \cdot n}{|n|} = \frac{2 + 2 + 15}{\sqrt{1 + 4 + 9}} = \frac{19}{\sqrt{14}}$$

$$\text{proj}_n m = \frac{m \cdot n}{|n|^2} n = \frac{19}{\left(\sqrt{14}\right)^2} \langle 1, 2, 3 \rangle$$

$$= \frac{19}{14} \langle 1, 2, 3 \rangle = \left\langle \frac{19}{14}, \frac{38}{14}, \frac{67}{14} \right\rangle$$