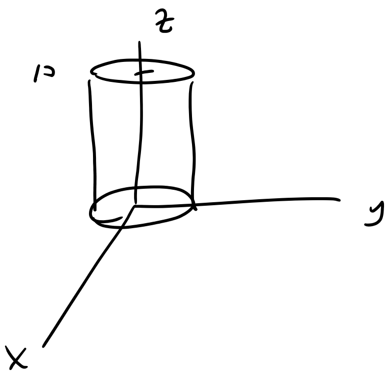
Section 16.9: Divergence Theorem

Divergence Theorem Let E be a simple solid region whose boundary surface has positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

Example: Let E be the solid bounded by $x^2 + y^2 = 25$, $z = 0$, and $z = 10$. Find the flux of the vector field $\mathbf{F} = \langle 1 + x, 2 + 3y, 2z + 5 \rangle$ over the boundary of the solid. Use positive orientation.

$\operatorname{div} F = 1 + 3 + 2 = 6$

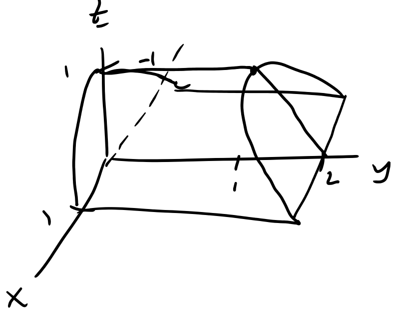


$$\iint_S F \cdot dS = \iiint_E \operatorname{div} F \, dV$$

$$= \iiint_E 6 \, dV = 6 \cdot \pi (5)^2 \cdot 10$$

$$= 1500\pi$$

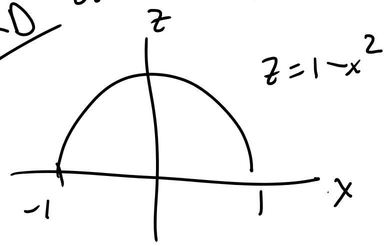
Example: Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \left\langle xy, y^2 + e^{xz^2}, \sin(xy) \right\rangle$ and S is the surface (with positive orientation) of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$.



$$\operatorname{div} F = y + 2y + 0 = 3y$$

$$\iiint_E 3y \, dV = \int_{x=-1}^{1} \int_{z=0}^{1-x^2} \int_{y=0}^{2-z} 3y \, dy \, dz \, dx$$

Region D on xz plane



Left $y = 0$

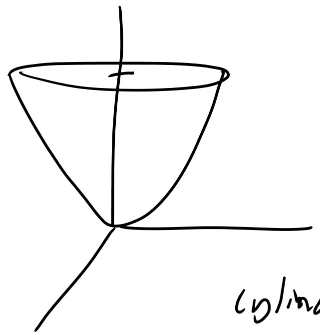
Right $y = 2 - z$

$$= \cdots = \frac{184}{5}$$

Example: Let S be the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$. Let $\mathbf{F} = \langle xz, yz, 3z^2 \rangle$. Use positive orientation.

Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where S is the boundary of the solid.

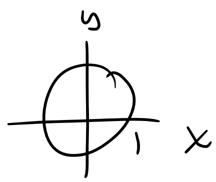
$\operatorname{div} F = z + z + 6z = 8z$



$$\iiint_E 8z \, dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=r^2}^{1} 8z \cdot r \, dz\, dr\, d\theta$$

$= \dots = \frac{8\pi}{3}$

cylindrical



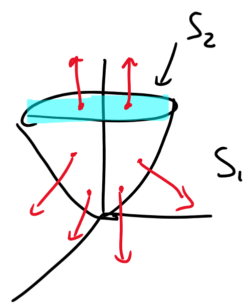
Top $z = 1$

Bottom $z = x^2 + y^2 = r^2$

not a closed surface

Example: Let S_1 be the surface of the paraboloid $z = x^2 + y^2$ for $0 \le z \le 1$ with downward orientation. Let $\mathbf{F} = \langle xz, yz, 3z^2 \rangle$.

Compute $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S_1}$



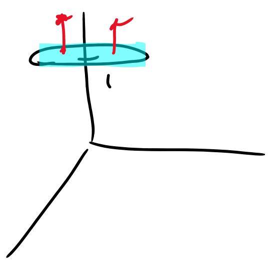
If I add S_2 which is the disk at $z=1$ with upward orientation

now we can use the divergence Thm on the surface of the object $S = S_1 + S_2$ to find $\iint_S F \cdot dS$

Know from last example $\iint_S F \cdot dS = \frac{8\pi}{3}$

Since we added S_2 we need to subtract $\iint_{S_2} F \cdot dS_2$

S_2



S_2

$x = x$
$y = y$
$z = 1$

cross product
$\langle 0, 0, 1 \rangle$ ✓

$F = \langle xz, yz, 3z^2 \rangle$
$= \langle x, y, 3 \rangle$

$$\iint_{S_2} F \cdot dS_2 = \iint_D F \cdot \text{cross product } dA = \iint_D 3 \, dA = 3\pi(1)^2 = 3\pi$$

final Answer $= \frac{8\pi}{3} - 3\pi$