Section 16.7: Surface Integrals

Definition: If S is parametrized by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, then the **surface integral of f over the surface S** is

$$\iint_S \underline{\underline{f(x,y,z)}}\,dS = \iint_D \underline{f(\mathbf{r}(u,v))}\,\underline{\underline{|\mathbf{r_u} \times \mathbf{r_v}|}}\,dA$$

where D is a region in the $uv-$plane.

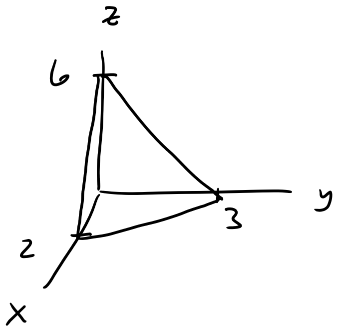
Application: If the function is the density at the points of the surface then the surface integral over S computes the mass of the surface.

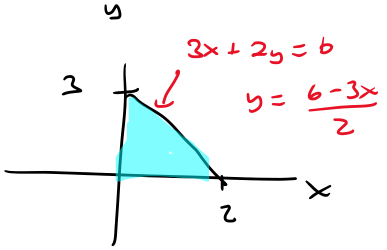
mass: $m = \iint_S \underline{\underline{\rho(x,y,z)}}\,dS$

Center of mass:

$$\overline{x} = \frac{1}{m}\iint_S x\rho(x,y,z)dS, \quad \overline{y} = \frac{1}{m}\iint_S y\rho(x,y,z)dS, \quad \overline{z} = \frac{1}{m}\iint_S z\rho(x,y,z)dS$$

Example: Evaluate $\iint_S xz\,dS$ where S is the part of the plane $3x + 2y + z = 6$ in the first octant.





$x = x$

$y = y$

$z = 6 - 3x - 2y$

$r(x,y) = \langle x, y, 6-3x-2y \rangle$

$\text{cross product} = \langle \overset{-f_x}{-(-3)}, \overset{-f_y}{-(-2)}, \overset{1}{1} \rangle$

$= \langle 3, 2, 1 \rangle$

$|\text{cross product}| = \sqrt{9+4+1} = \sqrt{14}$

$$\iint_S xz\,dS = \iint_D x(6-3x-2y)\cdot\sqrt{14}\,dA = \int_{x=0}^{2}\int_{y=0}^{\frac{6-3x}{2}} (6x-3x^2-2xy)\sqrt{14}\,dy\,dx$$

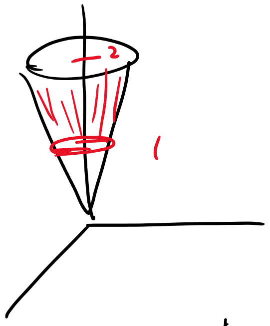
$= \ldots\ldots = 3\sqrt{14}$

Pg 3

Example: Compute $\iint\limits_S y^2z^2dS$ where S is the part of the cone $z = \sqrt{x^2+y^2}$ between the planes $z = 1$ and $z = 2$.

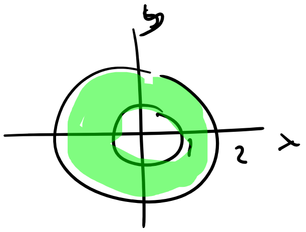
$1 = \sqrt{x^2+y^2}$

$1^2 = x^2+y^2$



$x = r\cos\theta$
$y = r\sin\theta$
$z = \sqrt{r^2} = r$

$r(r,\theta) = \langle r\cos\theta,\ r\sin\theta,\ r\rangle$



$$r_r \times r_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -r\cos\theta, -r\sin\theta, r\rangle$$

$$|r_r \times r_\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{r^2+r^2} = \sqrt{2r^2} = r\sqrt{2}$$

$$\iint\limits_S y^2z^2dS = \int_{\theta=0}^{2\pi}\int_{r=1}^{2} r^2\sin^2(\theta)\ r^2\ \overbrace{r\sqrt{2}}\ drd\theta = \ldots \frac{21\pi\sqrt{2}}{2}$$

Pg 3b

Example: Compute $\iint\limits_S y^2z^2dS$ where S is the part of the cone $z = \sqrt{x^2+y^2}$ between the planes $z = 1$ and $z = 2$.

Using $x = x \qquad y = y \qquad z = \sqrt{x^2+y^2}$

$$r(x,y) = \left\langle x,\ y,\ \sqrt{x^2+y^2} \right\rangle$$

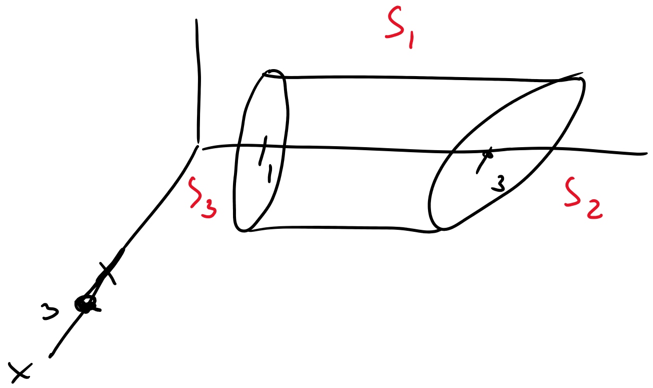
$$r_x \times r_y = \left\langle \overset{-f_x}{\frac{-x}{\sqrt{x^2+y^2}}},\ \overset{-f_y}{\frac{-y}{\sqrt{x^2+y^2}}},\ \overset{1}{1} \right\rangle$$

$$|r_x \times r_y| = \sqrt{\underbrace{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}_{} + 1} = \sqrt{2}$$

$$\iint\limits_S y^2z^2dS = \iint\limits_D y^2(\sqrt{x^2+y^2})^2\sqrt{2}\ \ dA = \iint\limits_D y^2(x^2+y^2)\sqrt{2}\,dA$$

$$\int_{\theta=0}^{2\pi}\int_{r=1}^{2} r^2\sin^2(\theta)\ r^2\ \sqrt{2}\ \ r\ drd\theta = \ldots \frac{21\pi\sqrt{2}}{2}$$

Example: Compute $\iint_S xy\,dS$ where S is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 1$ and $\underline{x + y = 3}$



S_1 cylinder

S_2 slant edge (slant cut)

S_3 perpendicular cut.

Side 1

$x = 1\cos\theta$ $\qquad 0 \le \theta \le 2\pi$

$y = y$ $\qquad 1 \le y \le 3 - x = 3 - \cos\theta$

$z = 1\sin\theta$

$r(y,\theta) = \langle \cos\theta, y, \sin\theta \rangle$

$r_y = \langle 0, 1, 0 \rangle$

$r_\theta = \langle -\sin\theta, 0, \cos\theta \rangle$

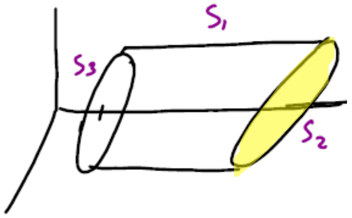
$$r_y \times r_\theta = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{vmatrix}$$

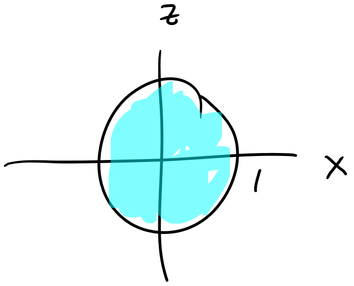
$$= \langle \cos\theta, 0, \sin\theta \rangle$$

$$|r_y \times r_\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$$

$$\iint_S xy\,dS = \int_{\theta=0}^{2\pi} \int_{y=1}^{3-\cos\theta} \cos\theta \cdot y \cdot 1 \, dy\, d\theta = -3\pi$$

Example: Compute $\iint_S xy\,dS$ where S is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 1$ and $x + y = 3$





side 2

$x = x$

$y = 3 - x$

$z = z$

cross product

$-f_x \,,\ 1 \,,\ -f_z$

$\langle -(-1),\ 1,\ 0 \rangle$

$c.p. = \langle 1, 1, 0 \rangle$

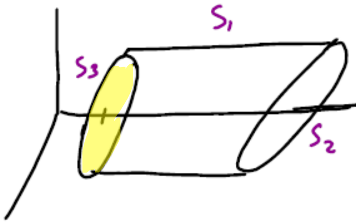
$|\text{cross product}| = \sqrt{1^2 + 1^2 + 0} = \sqrt{2}$

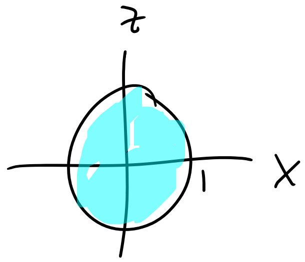
$$\iint_S xy\,dS = \iint_D x(3-x)\cdot\sqrt{2}\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\cos\theta(3 - r\cos\theta)\cdot\sqrt{2}\,r\,dr\,d\theta$$

$$= \ldots\ldots = -\frac{\sqrt{2}}{4}\pi$$

Pg 4c

Example: Compute $\iint_S xy\,dS$ where S is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 1$ and $x + y = 3$





Side 3

$x = x$

$y = 1$

$z = z$

$r(x,z) = \langle x, 1, z \rangle$

$-f_x, 1, -f_z$

$C.P. = \langle 0, 1, 0 \rangle$

$|C.P.| = \sqrt{1^2} = 1$

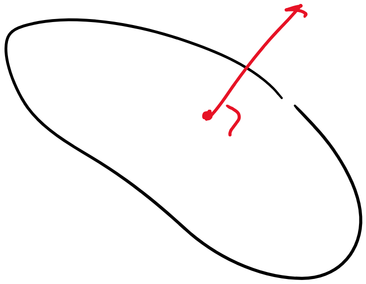
$$\iint_S xy\,dS = \iint_D x(1)\cdot 1\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\cos\theta \cdot r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2\cos\theta\,dr\,d\theta = \underbrace{\int_{\theta=0}^{2\pi}\cos\theta\,d\theta}\cdot\int_{r=0}^{1} r^2\,dr$$

$$= 0$$

$$\text{Answer} = -3\pi + \frac{-\sqrt{2}}{4}\pi + 0$$

Pg 5: surface integrals over vector fields



Let S be a surface parametrized by $\mathbf{r}(u,v)$. If S has a tangent plane at every point on S (except at any boundary points), then there are two unit normal vectors at every point.

$$\underbrace{\mathbf{n}_1 = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}} \text{ and } \mathbf{n}_2 = \underbrace{\frac{\mathbf{r}_v \times \mathbf{r}_u}{|\mathbf{r}_v \times \mathbf{r}_u|}}$$

The normal vector provides an orientation for S and S is called an **oriented surface**

For a surface defined by $z = g(x,y)$, then $\mathbf{n} = \underbrace{\frac{\langle -g_x, -g_y, 1\rangle}{\sqrt{1 + (g_x)^2 + (g_y)^2}}}$

Since the $\mathbf{k}$ component is positive, this gives the upward orientation of the surface.

Note: For a closed surface, a surface that is the boundary of a solid region(volume), **positive orientation** is where the normal vectors point outward from the region.

Pg 6

Definition: If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, the the **surface integral of F over S** is

$$\underline{\iint_S \mathbf{F}\cdot \underline{\underline{d\vec{\mathbf{S}}}} = \iint_S \mathbf{F}\cdot \underline{\underline{\mathbf{n}\; dS}}}$$

This integral is also called the **flux** of $\mathbf{F}$ across S.

Note: If S is parametrized by $\mathbf{r}(u,v)$, then $\mathbf{n} = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$

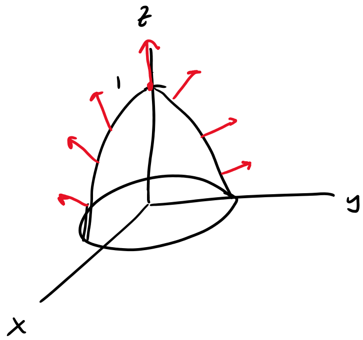
This gives $\underline{d\vec{\mathbf{S}} = \mathbf{n}\; dS} = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\; \underline{dS} = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\; \underline{|\mathbf{r}_u \times \mathbf{r}_v| dA} = (\mathbf{r}_u \times \mathbf{r}_v) dA$

Thus

$$\underline{\iint_S \mathbf{F}\cdot d\mathbf{S}} = \iint_S \mathbf{F}\cdot \mathbf{n}\; dS = \underline{\iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v) dA}$$

Note: choose the cross product that gives the correct orientation for the problem.

Example: Let S be the part of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane with upward orientation. Find the flux of $\mathbf{F} = \langle x, y, 3z \rangle$ across S.



Surface

$x = x$

$y = y$

$z = 1 - x^2 - y^2$

$r(x,y) = \langle x, y, 1 - x^2 - y^2 \rangle$

$$r_x \times r_y = \langle \overset{-f_x}{-(-2x)}, \overset{-f_y}{-(-2y)}, \overset{1}{1} \rangle$$

$$= \langle 2x, 2y, \underset{\uparrow}{1} \rangle$$

positive so upward orientation

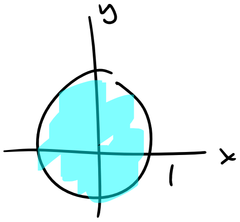
plugged in the surface

$$F = \langle x, y, 3z \rangle$$
$$= \langle x, y, 3(1 - x^2 - y^2) \rangle$$
$$= \langle x, y, 3 - 3x^2 - 3y^2 \rangle$$

$$\text{Flux} \quad \iint_S F \cdot dS = \iint_D \overset{F(r(x,y))}{\langle x, y, 3 - 3x^2 - 3y^2 \rangle} \cdot \overset{(r_x \times r_y)}{\langle 2x, 2y, 1 \rangle} \, dA$$

$$= \iint_D 2x^2 + 2y^2 + 3 - 3x^2 - 3y^2 \, dA = \iint_D 3 - x^2 - y^2 \, dA$$

Region D

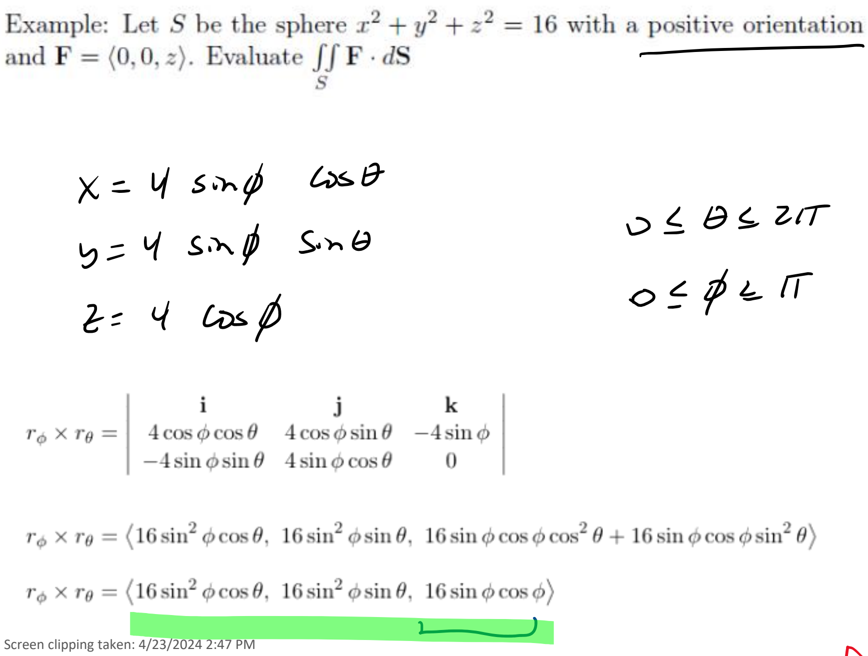


$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (3 - r^2) \cdot r \, dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} 1 \, d\theta \cdot \int_{r=0}^{1} (3r - r^3) \, dr = 2\pi \int_{r=0}^{1} 3r - r^3 \, dr$$

$$= \dots = \frac{5\pi}{2}$$

Pg 8



Example: Let S be the sphere $x^2+y^2+z^2=16$ with a positive orientation and $\mathbf{F}=\langle 0,0,z\rangle$. Evaluate $\iint\limits_S \mathbf{F}\cdot d\mathbf{S}$

closed surface and all normal vectors point outward.

$x = 4\sin\phi\cos\theta$

$y = 4\sin\phi\sin\theta$

$z = 4\cos\phi$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

$$r_\phi \times r_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4\cos\phi\cos\theta & 4\cos\phi\sin\theta & -4\sin\phi \\ -4\sin\phi\sin\theta & 4\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$r_\phi \times r_\theta = \langle 16\sin^2\phi\cos\theta,\ 16\sin^2\phi\sin\theta,\ 16\sin\phi\cos\phi\cos^2\theta + 16\sin\phi\cos\phi\sin^2\theta\rangle$$

$$r_\phi \times r_\theta = \langle 16\sin^2\phi\cos\theta,\ 16\sin^2\phi\sin\theta,\ \underbrace{16\sin\phi\cos\phi}\rangle$$

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correct direction

for $0 < \phi < \frac{\pi}{2}$

$\sin\phi > 0 \quad \cos\phi > 0$

z term is positive & thus upward orientation

$\pi/2 < \phi < \pi$

z term is negative

downward orientation

$F = \langle 0, 0, z\rangle$

$F = \langle 0, 0, 4\cos\phi\rangle$

$F \cdot (r_\phi \times r_\theta) = 4^3 \sin\phi\cos^2\phi$

$$\iint\limits_S F\cdot dS = \iint\limits_D F\cdot(r_\phi \times r_\theta)\,dA = \iint\limits_D 4^3\sin\phi\cos^2\phi\,dA$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} 4^3\sin\phi\cos^2\phi\,d\phi\,d\theta = \dots = \frac{4^4\pi}{3}$$

$$\int_{\theta=0} \int_{\phi=0} \qquad 3$$

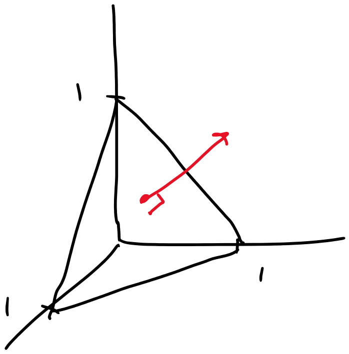
Example: Let S be the closed surface of a Tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, i.e the surface of the solid in the first octant that is formed by the plane $x+y+z=1$ and the three coordinate planes. Let $\mathbf{F} = \langle y, z-y, x\rangle$. and use positive orientation.
Evaluate $\iint\limits_S \mathbf{F}\cdot d\mathbf{S}$

$ax+by+cz=d$

$(1,0,0) \rightarrow a=d$

$(0,1,0) \rightarrow b=d$

$(0,0,1) \rightarrow c=d$



Sides

S_1 slant side (plane)

S_2 xy plane

S_3 xz plane

S_4 yz plane

Side 1

$x = x$

$y = y$

$z = 1-x-y$

$$r(x,y) = \langle x, y, 1-x-y\rangle$$

$$r_x \times r_y = \langle \overset{-f_x}{-(-1)}, \overset{-f_y}{-(-1)}, \overset{1}{1}\rangle$$

$$= \langle 1, 1, 1\rangle$$

Correct direction

$$F = \langle y, z-y, x\rangle$$

$$= \langle y, 1-x-y-y, x\rangle$$

$$= \langle y, 1-x-2y, x\rangle$$

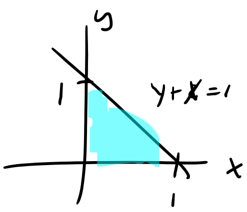
$$F\cdot(r_x\times r_y) = y + 1-x-2y + x$$

$$= 1-y$$

$$\iint\limits_{S_1} F\cdot dS_1 = \iint\limits_D F\cdot(r_x\times r_y)\, dA = \iint\limits_D 1-y\, dA$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} 1-y\, dy\, dx = \ldots = \frac{1}{3}$$

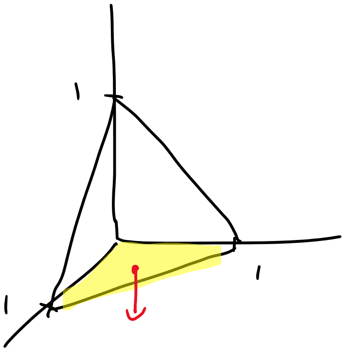
Region D



Example: Let S be the closed surface of a Tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, i.e the surface of the solid in the first octant that is formed by the plane $x+y+z=1$ and the three coordinate planes. Let $\mathbf{F} = \langle y, z-y, x\rangle$. and use positive orientation.
Evaluate $\iint\limits_S \mathbf{F}\cdot d\mathbf{S}$

$ax+by+cz=d$

$(1,0,0) \rightarrow a=d$
$(0,1,0) \rightarrow b=d$
$(0,0,1) \rightarrow c=d$



sides

S_1	slant side (plane)
S_2	xy plane
S_3	xz plane
S_4	yz plane

Side 2)

$x = x$
$y = y$
$z = 0$

$r(x,y) = \langle x, y, 0\rangle$

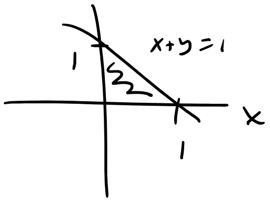
$r_x \times r_y = \langle -0, -0, 1\rangle$
$= \langle 0, 0, 1\rangle \leftarrow$ wrong direction.

use $\langle 0, 0, -1\rangle$

$F = \langle y, z-y, x\rangle$
$= \langle y, -y, x\rangle$

$F\cdot(\text{crossproduct}) = -x$

Region D



$$\iint\limits_{S_2} F\cdot dS_2 = \iint\limits_D -x \quad dA = \int_{x=0}^{1}\int_{y=0}^{1-x} -x\, dy\, dx$$

$$= \dots = \frac{-1}{6}$$

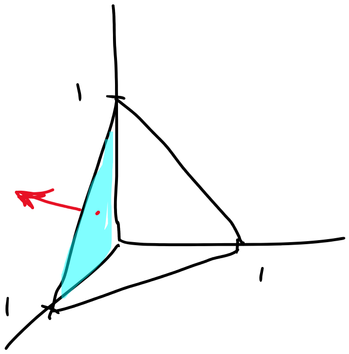
Example: Let S be the closed surface of a Tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, i.e the surface of the solid in the first octant that is formed by the plane $x+y+z=1$ and the three coordinate planes. Let $\mathbf{F} = \langle y, z-y, x\rangle$. and use positive orientation.
Evaluate $\iint\limits_S \mathbf{F}\cdot d\mathbf{S}$

$ax+by+cz=d$

$(1,0,0) \rightarrow a=d$

$(0,1,0) \rightarrow b=d$

$(0,0,1) \rightarrow c=d$



Sides

S_1 slant side (plane)

S_2 xy plane

S_3 xz plane

S_4 yz plane

Side 3

$x = x$

$y = 0$

$z = z$

$r(x,z) = \langle x, 0, z\rangle$

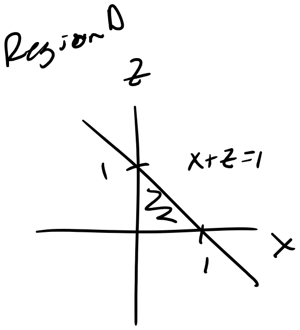
cross product $\langle 0, 1, 0\rangle$ wrong direction

use $\langle 0, -1, 0\rangle$

$$F = \langle y, z-y, x\rangle = \langle 0, z, x\rangle \Bigg\} \quad F \cdot \text{crossproduct} = -z$$

$$\iint\limits_{S_3} F\cdot dS_3 = \iint\limits_D -z \, dA$$

$$= \int_{x=0}^{1}\int_{z=0}^{1-x} -z \, dz\, dx = \dots = \frac{-1}{6}$$



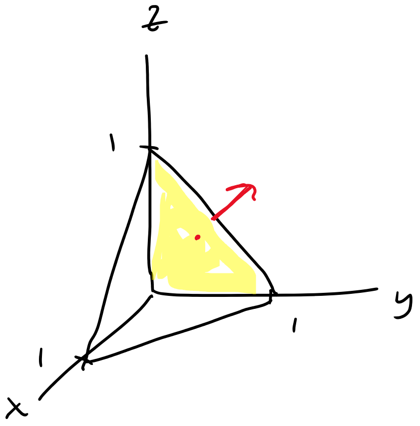
Example: Let S be the closed surface of a Tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, i.e the surface of the solid in the first octant that is formed by the plane $x+y+z=1$ and the three coordinate planes. Let $\mathbf{F}=\langle y, z-y, x\rangle$. and use positive orientation.
Evaluate $\iint\limits_S \mathbf{F}\cdot d\mathbf{S}$

$ax+by+cz=d$

$(1,0,0) \rightarrow a=d$

$(0,1,0) \rightarrow b=d$

$(0,0,1) \rightarrow c=d$



Sides

S_1 slant side (plane)

S_2 xy plane

S_3 xz plane

S_4 yz plane

Side 4

$x = 0$

$y = y$

$z = z$

$r(y,z) = \langle 0, y, z\rangle$

cross product $\langle 1,0,0\rangle$ wrong direction

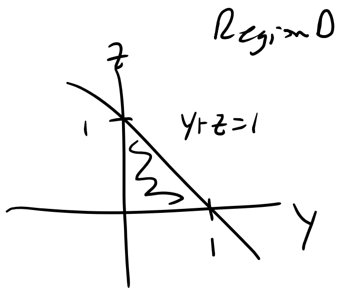
use $\langle -1,0,0\rangle$

$F = \langle y, z-y, x\rangle$

$= \langle y, z-y, 0\rangle$

$F \cdot$ cross product $= -y$

$$\iint\limits_{S_4} F\cdot dS_4 = \iint\limits_D -y \; dA = \int_{y=0}^{1}\int_{z=0}^{1-y} -y \, dz \, dy = \cdots = \frac{-1}{6}$$



Answer $\iint\limits_S F\cdot dS = \frac{1}{3} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6} = \frac{-1}{6}$